Three Looks at Instantons in F-theory

– New Insights from Anomaly Inflow, String Junctions and Heterotic Duality –

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ABSTRACT: We discuss the physics of zero modes of ED3/M5 instantons at strong coupling from three different viewpoints. Motivated by an anomaly inflow argument, we give a prescription for describing neutral instanton modes in terms of string junctions, unifying the language with that used for charged modes. We proceed to discuss the physics of charged modes as we move between different points in the moduli space of F-theory compactified on $K3$. In particular, we show how, in going from the E_6^3 point to the $SO(8)^4$ point, the structure of $SO(8)$ zero modes arises from a non-trivial mixing of massless $\mathbf{27}$'s of E_6 with massive modes stretching between different E_6 stacks. We observe a similar mixing in going from $SO(8)^4$ to E_6^3. Finally, we see how the zeroes of some exact worldsheet instanton superpotentials in heterotic backgrounds preserving E_6 symmetry admit a physical interpretation in terms of low energy physics. We also discuss the behavior of the dual F-theory compactification as the superpotential approaches a zero. An interesting observation is that in the examples we study some of the zeroes of the superpotential correspond to points of E_8 enhancement in the worldvolume of the dual $M5$ instanton, and more generally from enhancements of the singularity over the worldvolume of the instanton.

Contents

1   Introduction

Over the last few years there has been much focus on euclidean D3/D2 instantons[1] in weakly coupled type IIB/IIA string theory, which can generate superpotential corrections involving chiral matter fields that are forbidden by global U(1) selection rules in string perturbation theory [5–7]. Phenomenologically this is very important, as instantons can generate the $10\,10\,5_H$ top-quark Yukawa coupling in Georgi-Glashow GUTs [8] as well as a Majorana neutrino mass [5, 6], both of which are *always* forbidden in perturbation theory.

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[1]For a comprehensive review of D-instantons, see [1]. For reviews of intersecting D-branes in type II, see [2, 3], and for an introduction to both D-instantons and intersecting branes, see [4].

It is also well known that D-instanton effects can play a role in moduli stabilization, as used in [9] for the stabilization of Kähler moduli, for example.

Given the recent progress in understanding non-perturbative effects in weakly coupled type II, it is natural to investigate their relation to similar effects in F-theory [10], which contains type IIB as a weak coupling limit [11]. F-theory gives a nice framework for constructing semi-realistic GUT models [12, 13] in a way that takes advantage of both the local constructions of particle physics offered by intersecting braneworlds and the exceptional gauge symmetries common in the heterotic string. There has been much follow up work, both on important subtleties and model building [14–23]. An advantage of this framework is that one can study important aspects of particle physics such as the appearance of chiral matter and Yukawa couplings by studying codimension two and three singularities on the GUT 7-brane.

That one can learn much about the particle physics of an F-theory compactification by studying local geometry around the GUT brane does not mean that global effects are unimportant, however. The GUT 7-brane is in fact only one divisor in the Calabi-Yau fourfold base B_3, and effects which influence the physics of supersymmetry breaking and moduli stabilization (for example) can occur away from the GUT brane in B_3. Moreover, many three generation local models do not even admit a global embedding [17]. For these and other reasons, there have been a number of works studying the physics of global F-theory GUTs [22, 24–31], including some explicit global models where the elliptic Calabi-Yau fourfold is a CICY in a toric variety.

The physics of instantons in F-theory is one example which depends heavily on the global geometry, as they wrap divisors which are generically different from the GUT brane divisor and sometimes do not even intersect it. Though instantons in F-theory are not "needed" to generate the $10\,10\,5_H$ Yukawa coupling as they are in type IIB, due to the existence of that Yukawa coupling at a point of E_6 enhancement, they nevertheless still generate superpotential corrections, play a crucial role in the stabilization of Kähler moduli, and can account for large hierarchies due to exponential suppression. From a more pragmatic point of view, one must account for them simply because they are there and will affect the physics.

In studying instanton effects in F-theory, it often proves useful to understand them via duality to M-theory and the heterotic string, and also in the type IIB weak coupling limit. For example, via duality with M-theory on a elliptic Calabi-Yau fourfold in the limit of vanishing fiber, it was shown in [32] that in an F-theory compactification to four dimensions a necessary condition for M5-brane instantons to correct the superpotential is that the divisor D of the M5 instanton satisfies $h^0(D, \mathcal{O}_D) = 1$ and all other $h^i(D, \mathcal{O}_D) = 0$. In addition, the instanton must be a "vertical" M5 brane, meaning that D wraps the total space of the fibration over a divisor $\tilde{D}$ of the base B_3. One can relate the description in terms of the M5 brane in the four-fold (the M-theory description of the system) with a description purely in terms of an euclidean D3 on the base B_3 (the IIB description). In particular the relation between the counting of neutral zero modes was given in [33]. For this reason, one often refers to ED3/M5 instantons in F-theory.

Under heterotic / F-theory duality, some ED3/M5 instantons in F-theory dualize to

heterotic worldsheet instantons, while others dualize to NS5-brane instantons. More precisely, given a compactification of F-theory on an elliptically fibered $K3$, dual to a T^2 compactification of the heterotic string, M5 branes wrapping the whole $K3$ dualize to worldsheet instantons, while those wrapping the fiber but not the base of the $K3$ dualize to NS5 branes on the heterotic side. This extends naturally to the $K3$ fibered backgrounds we will be considering in this paper, accordingly we will be focusing on M5 branes wrapping the whole $K3$ fiber. Some aspects of ED3/M5 instantons in F-theory were studied from the point of view of heterotic worldsheet instantons in [34]. Other works on instantons in F-theory include the lift of an ED3 instanton which generates the $10\,10\,5_H$ coupling to a global F-theory model [25], instantons in local F-theory models [35, 36], and the use of instanton flux [37] to alleviate the generic tension between moduli stabilization and chirality [38].

Despite this progress, there are still a number of issues regarding instantons in F-theory which must be addressed. One issue is that a complete understanding of the chirality inducing G-flux, inherently in F-theory without reference to a heterotic dual, is still lacking, though a proposal has been made in [39, 40]. This issue, which faces F-theory compactifications generically, also has strong implications for instantons. Aside from these issue, though, there are very basic questions which still need to be addressed. For example, given a globally defined F-theory compactification with G-flux

- What are the "charged" zero modes stretching between the instanton and 7-branes, and how do they relate to the well-understood charged modes in type IIB?

- How does one compute the superpotential correction due to an ED3/M5 instanton?

In weakly coupled type II compactifications with a CFT description (such as models compactified on a toroidal orbifold) both of these questions have concrete answers. The charged zero modes are represented by vertex operators obtained by the quantization of open strings stretching between and ED2(ED3) instanton and a D6(D7) brane in type IIA(IIB). Their superpotential corrections, if any, can be calculated according to the instanton calculus of [5], and couplings between charged instanton zero modes and chiral matter fields can be computed via disc diagrams in the CFT. Of course, F-theory compactifications generically involve highly curved manifolds and strong coupling, so that they do not admit at CFT description. Therefore, one must address and answer these questions using a different formalism.

In this paper we aim to shed some light on this questions by using a multi-pronged approach. In section 2 we will reinterpret an observation of [41] regarding anomaly inflow towards orientifolds in the context of instantons in F-theory. This will give us a way of understanding neutral zero modes in terms of string junctions, and in particular we give a strong coupling re-interpretation of the familiar θ mode as a particular junction with prongs on both orientifold components. We proceed to study charged zero modes in section 3 using string junctions techniques. We will see that modes that are massless in certain regions of moduli space can come from rather complicated multi-pronged strings in other regions. These multi-pronged strings often stretch between distant 7-branes, and therefore can

correspond to massive BPS states. We illustrate this discussion in a particularly simple family of configurations that interpolates between E_6^3 gauge symmetry and $SO(8)^4$. Finally, in section 4 we analyze the non-perturbative physics of certain heterotic backgrounds with E_6 symmetry at low energies. The exact dependence on the vector bundle moduli of the superpotential due to a particular worldsheet instanton was computed in [42, 43], and we give a physical understanding of this dependence in terms of low energy effective field theory. By analyzing the behavior of the dual F-theory compactification as we reach the zeroes of the superpotential, we observe some interesting features of the physics of M5 instantons in the non-perturbative regime. Certain points in moduli space corroborate what is expect from weakly coupled type II, where a zero of the superpotential correction due to an ED3 instanton corresponds to the appearance of extra light matter in the spectrum.

Note added: As we were readying this paper for publication, we received a draft copy of [40], which has some overlap with the discussion in section 4. We thank the authors of [40] for letting us know of their work prior to publication.

2 Anomaly inflow and a new description of neutral zero modes

We will start our discussion by giving in this section a formulation of neutral zero modes of rigid $O(1)$ instantons (that is, the modes commonly known in the recent instanton literature as θ modes) valid in regions of moduli space arbitrarily far away from weak coupling.

In the conformal field theory approach to instantons, neutral and charged zero modes arise from fundamentally different objects: neutral zero modes arise from strings going from the instanton to itself, while charged modes arise from strings going from the instanton to space-time filling D7 branes intersecting the instanton. As we will see in section 2.1, analyzing the system in detail reveals that θ modes localize in the intersections of the instanton with the background orientifolds. This fact partially blurs the strong split one observes in the perturbative approach between both kinds of zero modes, since now both localize around defects in the worldvolume theory on the instanton.

In going to strong coupling the orientifold decomposes into (p,q)-7 branes, and close to each of the components of the orientifold the system is a $SL(2,\mathbb{Z})$ transform of the one giving a charged mode. So our next task is to explain why this mutually non-local pair of charged modes gives rise to something that at large distances looks like an uncharged mode. As we will see in section 2.2, doing this properly blurs even further the distinction between both kinds of modes.

Even if non-perturbatively both neutral and charged modes admit a unified description, they are still fundamentally different at low energies (they transform differently under the Lorentz group of spacetime, for one). In section 2.3 we provide a criterion for determining microscopically which kind of zero mode one is dealing with.

We will come back to study the behavior of charged modes from a non-perturbative perspective in section 3, but for the rest of this section we will focus mainly on neutral modes.

Let us consider F-theory compactified on $K3$, starting from situations close to weak coupling. We will take the euclidean D3 to wrap the $\mathbb{P}^1$ base of the $K3$, and two extra directions transverse to the K3.[2] In this setting the worldvolume theory of the instanton is four dimensional topologically twisted $\mathcal{N} = 4$ $U(1)$ SYM, compactified on an $\mathbb{P}^1$, and in the presence of 24 string-like defects which are point-like on the $\mathbb{P}^1$. Close to weak coupling the 24 defects split naturally into 16 mutually local defects (the D7 branes), and 4 pairs of mutually non-local defects (the 4 $O7$-planes).

In order to keep the discussion as simple as possible, we will take a decompactification limit of the $\mathbb{P}^1$ in which the curvature vanishes, while keeping the 24 branes at a finite distance. Our discussion will only involve topological quantities, which are robust under this deformation. In this flat limit the twisting becomes unimportant, and the worldvolume theory of the instanton becomes 4d $\mathcal{N} = 4$ $U(1)$ SYM in the presence of string-like defects.

2.1 $1/4$ of a θ mode per orientifold

We will like to understand how to describe instanton zero modes in such a background in a way valid away from weak coupling. In order to better motivate our later results we will need to take a small detour and study anomaly inflow in our configuration. This analysis has been done with a different motivation (AdS/CFT) in [41]; in this section we review their discussion and present some numeric results that support and illustrate their result. We will reinterpret the discussion of [41] in sections 2.2 and 2.3.

The basic puzzle that [41] addresses can be summarized as follows: take the euclidean D3 brane to be wrapping a 4-manifold X (in our case X is a copy of flat space $\mathbb{R}^4$). There is also a $O7^-$ plane wrapping a divisor of X (i.e. a $\mathbb{R}^2$ subspace of $\mathbb{R}^2$ in the flat case), and 6 extra dimensions. We denote by Y the total space wrapped by the $O7^-$.[3]

The Chern-Simons coupling on the $O7^-$ plane is given by [45–49]:

$$S_{cs}^{O7^-} = \int_Y C \wedge \sqrt{\frac{\hat{L}(TY/4)}{\hat{L}(NY/4)}}\,, \tag{2.1}$$

where C denotes the formal sum of RR forms, and TY and NY denote respectively the tangent bundle to Y and the normal bundle to Y inside the ambient Calabi-Yau. Given a vector bundle E, $\hat{L}(E)$ denotes the Hirzebruch genus of E, defined by:

$$\hat{L}(E) = \prod \frac{x}{\tanh x} = 1 + \frac{1}{3}(c_1^2(E) - 2c_2(E)) + \dots\,, \tag{2.2}$$

and as it is conventional we have denoted by x the components of E under splitting. The

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Chern-Simons coupling on the D3 is also well known, and it is given by:

$$S_{cs}^{D3} = \int_X C \wedge \text{ch}(V) \wedge \sqrt{\frac{\hat{A}(TX)}{\hat{A}(NX)}}, \tag{2.3}$$

with the same conventions as before for C, TX and NX; and $\hat{A}$ being the A-roof (or Dirac) genus, given to the first few orders by:

$$\hat{A}(E) = \prod \frac{x_i/2}{\sinh(x_i/2)} = 1 - \frac{1}{24}(c_1^2(E) - 2c_2(E)) + \dots \tag{2.4}$$

In order to be general we have also included a factor of $\text{ch}(V)$, where V is a possible vector bundle on the stack of D3 branes. In our case we have a single D3 with gauge group $\mathbb{Z}_2$, so we take $V = 1$ in the following.

Given (2.1) and (2.3), a standard anomaly inflow argument (see [50] for a review) shows that there is an anomaly localized on the 2d intersection between the $O7^-$ and the $D3$. The associated 4-form anomaly polynomial is given by:

$$I_{inflow} \sim p_1(TX) - p_1(NX) - \frac{1}{2}(p_1(TY) - p_1(NY)), \tag{2.5}$$

where we have omitted some numerical factors which are irrelevant at this level of the argument, and for conciseness we have introduced the first Pontryagin class $p_1(E) = c_1^2(E) - 2c_2(E)$.

The puzzle is now evident: the fact that (2.5) is non-vanishing indicates that consistency of the background requires some chiral degrees of freedom to live in the intersection in order to cancel the anomaly. Nevertheless, there are no obvious candidates in the form of strings between the D3 and the $O7^-$ plane. By this we mean simply that there is no twisted sector for orientifolds. Strings going from the instanton to the $O7^-$ are interpreted as unoriented D3-D3 strings, and if they are the source for the chiral modes, the analysis is necessarily somewhat subtle: 3-3 states are associated with states in the 4d theory on the instanton which survive the orientifold projection, and one would think that states in the 4d theory look non-chiral from the 2d point of view.

Indeed, the answer that [41] suggests, and convincingly argues in a beautiful analysis, is that the required chiral matter comes from a zero mode of the gaugino on the euclidean D3. In our particular context, this means that the θ mode localizes on the intersections of the instanton with the background orientifolds. To our knowledge, this is a new (and to us, surprising) observation in the context of instanton physics, and as we will see in the rest of this section it provides the key to understanding instanton zero modes at arbitrarily strong coupling.

Before going into the consequences of this observation for instanton physics, and since the claim may be a bit surprising, let us present some simple numerical results that clearly illustrate the phenomenon.

Let us consider F-theory compactified on an elliptically fibered $K3$ with section. We

wrap the euclidean D3 on the $\mathbb{P}^1$ base of the $K3$, and thus the D7 and $O7^-$ planes appear as 24 point-like defects on the $\mathbb{P}^1$. We want to stay close to weak coupling for ease of interpretation, so let us parameterize the Weierstrass parameters of the elliptic fibration in Sen's form [11, 51]:

$$f = -3h(z)^2 + \epsilon\eta(z) \tag{2.6}$$

$$g = -2h(z)^3 + \epsilon h(z)\eta(z) - \frac{1}{12}\epsilon^2\chi(z) \tag{2.7}$$

with z the complex coordinate on $\mathbb{P}^1$, ϵ is an arbitrary number parameterizing how close we are to weak coupling (we take $\epsilon = 10^{-3}$), and $h(z)$, $\eta(z)$ and $\chi(z)$ arbitrary functions. In terms of this parameterization, the discriminant is given by

$$\begin{aligned}
\Delta &= 4f^3 + 27g^2 \\
&= 9\epsilon^2 h^2(h\chi - \eta^2) - \frac{1}{2}\epsilon^3\eta(9h\chi - 8\eta^2)\,,
\end{aligned} \tag{2.8}$$

where we have dropped the explicit dependence on z for readability. The weak coupling limit is given by $\epsilon \to 0$, in which case we have $\Delta \sim h^2(h\chi - \eta^2)$. The 7 branes are given by the roots of Δ, are they are thus located at $h(z) = 0$ and $(h\chi - \eta^2) = 0$. A monodromy analysis shows that the first set of roots corresponds to the location of $O7^-$ planes, and the second set to the location of D7 branes.

We take these arbitrary functions to be:

$$\begin{aligned}
h(z) &= \prod_{n=1}^{4}(z - h_n) \\
\chi(z) &= 0 \\
\eta(z) &= \prod_{n=1}^{8}(z - \eta_n)
\end{aligned} \tag{2.9}$$

with

$$\eta_n = 1.3\,e^{2\pi in/8} \qquad ; \qquad h_n = \{0.6 + 0.35i, 0.35 - i/2, -0.25 - 0.45i, -1/2 + i/2\} \tag{2.10}$$

The exact numeric values are inessential, the basic feature of this choice being that the orientifold planes get distributed in the corners of a (slightly deformed) square, with positions given roughly by the zeroes of $h(z)$.[4] The 16 D7 branes split into 8 pairs of branes, each pair located at a zero of $\eta(z)$, which we choose in (2.10) to be arranged concentrically. We show a plot of the resulting discriminant in figure 1a.

We now want to see how the θ zero mode localizes near the orientifold planes. The solution for the wavefunction of θ on $\mathbb{P}^1$ is given in [41] in terms of a function $b(z)$ given

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[4]We say "roughly" since the identification of $h(z) = 0$ with orientifolds is only valid if we disregard the second term in (2.8) — which given our small value for ϵ is a good approximation — but the plots in the text are exact in ϵ. Similar remarks apply to the position of the D7 branes.
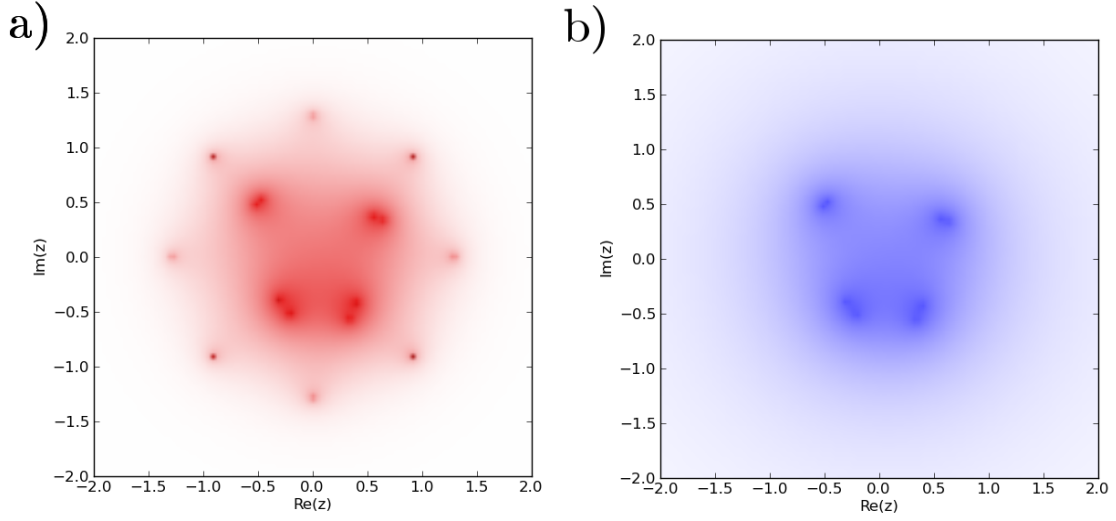
Figure 1. a) Discriminant Δ for the choice of h, η, χ in (2.9), (2.10). b) Corresponding $b(z)$ for θ, we are plotting $|b(z)|$.

by:

$$b(z) = \frac{\eta(\tau(z))}{\Delta^{1/24}}, \tag{2.11}$$

with Δ the discriminant of the elliptic fibration. The coupling $\tau(z)$ can be determined in terms of hypergeometric functions using the expression for the inverse of Klein's J-invariant:

$$\tau(z) = J^{-1}(j(z)/1728) \tag{2.12}$$

where

$$J^{-1}(\lambda) = \frac{i(r(\lambda) - s(\lambda))}{r(\lambda) + s(\lambda)} \tag{2.13}$$

and we have introduced

$$\begin{aligned}
r(\lambda) &= \Gamma\left(\frac{5}{12}\right)^2 {}_2F_1\left(\frac{1}{12}, \frac{1}{12}; \frac{1}{2}; 1 - \lambda\right) \\
s(\lambda) &= 2(\sqrt{3} - 2)\Gamma\left(\frac{11}{12}\right)^2 \sqrt{\lambda - 1} \, {}_2F_1\left(\frac{7}{12}, \frac{7}{12}; \frac{3}{2}; 1 - \lambda\right)
\end{aligned} \tag{2.14}$$

and ${}_2F_1$ denotes the ordinary or Gaussian hypergeometric function. We have plotted the absolute value of $b(z)$ in figure 1b, where it is clear that the resulting non-trivial behavior of the wavefunction localizes around the orientifold planes.

While figure 1b shows the localization of the zero mode structure in regions close to the orientifolds, the full story is slightly more complicated. The wavefunction obtained by

[41] can be written as:[5]

$$\Psi(z) = e^{-i \arg(b(z))} \tag{2.15}$$

where $\arg(e^{i\alpha}) = \alpha$. Let us analyze first the monodromy around a D7. As we go around a D7 brane we have that $\tau \to \tau + 1$, which sends $\eta(\tau) \to \eta(\tau + 1) = e^{\frac{\pi i}{12}}\eta(\tau)$. We are also going around a single zero of the discriminant, which thus sends $\Delta^{\frac{1}{24}} \to e^{\frac{\pi i}{12}}\Delta^{\frac{1}{24}}$. From the form of $b(z)$ in (2.11), we then immediately see that $b(z)$, and thus $\Psi(z)$ is invariant as we go around a D7 brane. On the other hand, as we go around an $O7^-$ plane, we have that $\tau \to \tau - 4$. Correspondingly, we have that $\eta(\tau) \to \eta(\tau - 4) = e^{-\frac{\pi i}{3}}\eta(\tau)$. Since in going around an $O7^-$ we pick a double zero of Δ, we have that $\Delta^{\frac{1}{24}} \to e^{\frac{\pi i}{6}}\Delta^{\frac{1}{24}}$. In terms of the wavefunction we thus have that $\Psi(z) \to -i\Psi(z)$. (As a side remark, notice that this is the behavior one should expect from the action of $SL(2,\mathbb{Z})$ on the fermions of the $\mathcal{N} = 4$ theory, as derived in [52] and used in this same context in [44].) So we see that the non-trivial variation of the θ mode concentrates on the orientifold planes.

2.2 What flows

The results in the previous section can be given a very useful interpretation in the following way. Notice that in the case of ordinary D7 branes intersecting the instanton the anomaly inflow is canceled due to the zero modes localized at the D7 defect, i.e. the massless modes associated to the fundamental strings between the instanton and the D7 brane. The analogous candidate string at weak coupling for canceling the inflow to the orientifold defect (which, as we saw, is also where the θ mode localizes) are strings going from the instanton to the orientifold.

This is perhaps a little bit surprising to people familiar with instanton dynamics: usually we think of the θ mode as the gaugino of the $\mathcal{N} = 4$ theory on the instanton that survives the orientifold projection. As such, we are not used to thinking of it as a localized mode. Nevertheless, it is important to realize that zero modes of the gaugino on the worldvolume of the instanton, by their very nature, are extended objects, and can appear or disappear in the presence of defects. In the case of θ the counting does not change, but the profile of the zero mode does.

Strings from the instanton to the orientifold can be alternatively interpreted as the 3-3 strings mapped to themselves under the orientifold action, the statement above being in this language that θ modes come from invariant 3-3 strings.

Having identified the θ mode in the perturbative language, we would now like to turn on g_s. We encounter a small puzzle here: as we turn on the coupling the orientifold splits into two components, and there is no massless fundamental string anymore. This is analogous to what happens in Seiberg-Witten theory for the W boson [53, 54], and we propose that the resolution is precisely the same in our case: the fundamental string associated to the θ mode turns into a string junction. See figure 2.

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[5]The wavefunction (2.15) was derived using techniques valid only at weak coupling, and as such we can only trust it far away from the orientifolds. It will nevertheless suffice for showing localization.
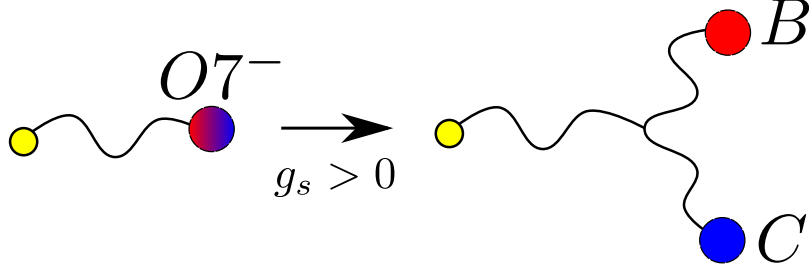
Figure 2. As the orientifold splits into its B, C components due to the effect of $D(-1)$ instantons, the orientifold invariant 3-3 string corresponding to the θ mode becomes a string junction. Note that the junction is formally the same as the one describing W bosons in the string realization of Seiberg-Witten.

The way to understand the fact that a massive string junction is associated to a zero mode is the following. Let us turn on an electric field on the instanton worldvolume in the directions parallel to the defect. This is slightly ill-defined in the strongly coupled region inside the orientifold, so let us postpone discussing what happens there momentarily, and work far away from the orientifold. Due to the monodromy of the axion around the orientifold, the electric field produces an electric current *towards* the defect. The derivation is elementary, and we review it here for convenience: in the presence of an axion $a(x)$, the $U(1)$ Yang-Mills Lagrangian looks like:

$$S_{U(1)} = \frac{-1}{g^2} \int F \wedge \star F + aF \wedge F + \dots, \tag{2.16}$$

where we have ignored terms irrelevant for our analysis. Integrating by parts, and taking the variation with respect to the $U(1)$ connection, we have:

$$d \star F = g^2 d(aF). \tag{2.17}$$

This implies that in the presence of a non-constant axion and a non-trivial background F we have a current vector $j = \star g^2 d(aF)$ for the electromagnetic field. In particular, choosing x^0, x^1 as the directions along the defect, and θ, r the normal directions (in polar coordinates), it is clear that choosing $F = E \, dx^0 \wedge x^1$, $a = \theta$ gives a current $j \sim dr$.

This current towards the string will violate charge conservation unless something happens at the string that takes away the charge. The "something" that happens is that generically there are chiral zero modes on the string which are charged under the bulk $U(1)$, so when the electric field is applied they flow along the string taking away the charge. These modes are localized in the sense that they have exponentially decaying wavefunctions, and we will refer to them as *carriers* in what follows. (See [55] for a careful study of the dynamics of the system using the same viewpoint that we are taking here.)

In our orientifold case, we also have axion monodromy, and thus current flowing into the orientifold. The modes that take away the current are the θ junctions.

We come now to an important point: the θ junctions exist as stable carriers when we

are far way from the orientifold. As we enter the region of strong coupling the junction becomes unstable, and decays into a couple of elementary constituents, going to each of the components of the orientifold. The analysis is identical to the familiar decay of the W boson at strong coupling in the Seiberg-Witten theory, and we omit repeating the well-known details of this process (see [56–58] for in-depth discussions from the field theory point of view, and [59, 60] for the string junction point of view). Seeing the same decay process inversely, this gives a natural explanation to the question of what happens to the putative charged zero modes associated to each component of the orientifold: locally there are indeed dyonic zero modes of the instanton, carried by the corresponding strings, but globally the carriers for the local zero modes form bound states which are purely electric, the θ junctions.

Let us give a couple of further arguments in favor of this proposal. First of all, a somewhat counter-intuitive fact is that the θ mode has charge under the $U(1)$ that exists locally on the instanton. More precisely, its charge is 2, in the conventions where an ordinary 3-7 string has charge 1. (The analogous statement in the Seiberg-Witten case is that the W boson has charge 2 in the conventions where the charge of the elementary flavor hypermultiplet is 1.) This is in fact in agreement with the fact that the resulting zero mode is uncharged under the $\mathbb{Z}_2$ symmetry remaining on the instanton after the orientifold action.

Also notice that the θ modes form a doublet under the $SU(2)_{\mathcal{R}}$ symmetry (in analogy with the Seiberg-Witten case). Together with the fact that the junction has double the $U(1)$ charge as an ordinary string, we see that the θ junctions carry precisely 4 times the charge of an elementary string carrier (an ordinary 3-7 string). This is in agreement with the fact that the monodromy of the axion around an orientifold is 4 units, as opposed to 1 unit for an ordinary D7 brane.

2.3 Neutral zero modes as vector multiplets

The previous proposal gave a very nice unified description in the language of string junctions of the origin of both neutral and charged zero modes of the instanton. One may wonder how to distinguish both cases in practice. The answer has already been implicitly given in the previous section, but let us be explicit about it here.

Take the string junction one is interested in analyzing. There will be an analogous junction in Seiberg-Witten theory. The resulting state will be either a hypermultiplet or a vector multiplet, this can be determined using standard techniques. The discussion in the previous section shows that we should understand hypers as giving rise to charged zero modes, while vectors give rise to neutral zero modes. The basic distinction is their behavior under $SU(2)_{\mathcal{R}}$, which we can identify as rotations in spacetime: vectors, just as the θ, transform as spinors, while hypers, such as the 3-7 strings, are singlets.

3 Charged modes as string junctions

In the previous section we argued that neutral zero modes of the instanton admit a very similar description to their charged counterparts, once we formulate the physics using string

junctions. In this section we continue our study of instanton zero modes and address the question of how do charged modes in strongly coupled limits of F-theory relate to their perturbative type IIB counterparts.

We would like, in particular, to clarify those aspects of instantons in F-theory which are qualitatively different from the physics of instantons in type IIB, such as those arising in the presence of exceptional 7-brane singularities in the instanton worldvolume. Whatever the correct formalism is for charged instanton corrections in F-theory, it should recover the known type IIB results in Sen's limit, so one might hope to see this directly by studying instanton effects as one moves in the complex structure moduli space of the F-theory compactification.

Our approach is to study the structure of charged instanton zero modes, realized as string junctions, as one moves in the complex structure moduli space of an F-theory compactification. We will present a particular path in the moduli space of F-theory on $K3$ and explicitly study the fate of $\mathbf{27}$'s of E_6 upon movement in moduli space to the type IIB limit. We will also study an $\mathbf{8}_v$ of $SO(8)$ in type IIB and will show that, while some of the junctions in the $\mathbf{8}_v$ contribute to $\mathbf{27}$'s of E_6 elsewhere in moduli space, others are massive BPS states with prongs stretching between distant E_6 singularities. It will become clear that this type of behavior is not uncommon upon movement in moduli space.

3.1   A review of 7-branes and junctions

Let us review those aspects of 7-branes and junctions necessary for our discussion of charged instanton zero modes. We will make some definitions and present some constant coupling results relevant for our study. Our conventions are mostly taken from [61].

Much of the interesting physics arising from F-theory compactifications descend from the fact that F-theory allows for more generic 7-branes than type IIB, with a notable feature being that the 7-branes have non-trivial monodromy. The monodromy upon counterclockwise crossing of a (downwards-directed) branch cut[6] is given by

$$\mathcal{M} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2, \mathbb{Z}) \tag{3.1}$$

and acts on the axio-dilaton $\tau = C_0 + \frac{i}{g_s}$ as

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}. \tag{3.2}$$

A (p, q) 7-brane is such a 7-brane, where the monodromy matrix is given by:

$$\mathcal{M}_{(p,q)} = \begin{pmatrix} 1 + pq & -p^2 \\ q^2 & 1 - pq \end{pmatrix}. \tag{3.3}$$

An important fact is that 7-branes with exceptional gauge symmetry (as well as others) can not be represented by a single (p, q) 7-brane, but instead can be represented by a number

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[6]For an in depth discussion of 7-branes, branch cuts, and junctions, see [62].

of (p, q) 7-branes. A convenient choice of basis for (p, q) 7-branes that give rise to the ADE groups is:

$$\mathcal{A} = \mathcal{M}_{(1,0)} = \begin{pmatrix} 1 & -1 \\ 0 & 1 \end{pmatrix} \qquad \mathcal{B} = \mathcal{M}_{(1,-1)} = \begin{pmatrix} 0 & -1 \\ 1 & 2 \end{pmatrix} \qquad \mathcal{C} = \mathcal{M}_{(1,1)} = \begin{pmatrix} 2 & -1 \\ 1 & 0 \end{pmatrix}. \quad (3.4)$$

In the type IIB picture, an A-brane is a $D7$ brane and a B and C brane together give rise to an $O7$-plane. This is the famous splitting of an $O7$-plane in F-theory. It splits into two branes, a B and a C, with separation set by e^{-1/g_s}.

As an example, a 7-brane with $SO(8)$ singularity can be represented as
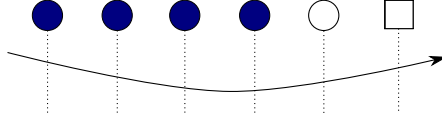
where solid circles, hollow circles, and hollow squares represent A branes, B branes, and C branes, respectively. The dotted lines represent the downward directed brane cuts, and the arrow represents the "direction" of the monodromy in the conventions we are using. The monodromy matrix is

$$\mathcal{M}_{SO(8)} = \mathcal{C}\mathcal{B}\mathcal{A}^4 = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}. \quad (3.5)$$

The monodromy matrices act to the right, so the order of matrices is the opposite of the left-right ordering of branes. We recognize this convention is somewhat confusing, but it is fairly standard in the literature and so we adopt it.[7]

Similarly, the E_6 singularity can be obtained from a stack with left-right ordering $AAAAABCC$, with monodromy

$$\mathcal{M}_{E_6} = \mathcal{C}^2\mathcal{B}\mathcal{A}^5 = \begin{pmatrix} -2 & 3 \\ -1 & 1 \end{pmatrix}. \quad (3.6)$$

This singularity leaves constant the coupling satisfying:

$$\tau = \frac{-2\tau + 3}{-\tau + 1}, \quad (3.7)$$

which has the solutions $\tau = e^{\pm\frac{\pi i}{3}} + 1$. Only the plus sign is physical, since $Im(\tau) = \frac{1}{g_s}$ needs to be positive. Thus, using these conventions, the monodromy associated with an E_6 singularity leaves $\tau = e^{\frac{\pi i}{3}} + 1$ invariant. We will be studying constant coupling configurations with E_6 singularities, which were introduced in [63]. The conventions of [63] are such that $\tau = e^{\frac{\pi i}{3}}$, which differs from our case by 1, i.e. a T-monodromy. However, we

---

[7]We will attempt to minimize the resulting confusion by denoting the branes themselves in italic script: $A, B, C, \ldots$, while the corresponding monodromy matrices will be written in calligraphic script: $\mathcal{A}, \mathcal{B}, \mathcal{C}, \ldots$

stick with these conventions throughout, as they agree with [61] , the methods of which we will use extensively.

It will be of importance four our analysis that the monodromy associated with an $E6$ 7-brane can be written as

$$\mathcal{M}_{E_6} = \mathcal{C}^2 \mathcal{B} \mathcal{A}^5 = \mathcal{C} \, \mathcal{M}_{D_4} \mathcal{A}, \tag{3.8}$$

corresponding to a left-right ordering $AAAAABCC$, where the monodromy of an $SO(8)$ 7-brane sits in the middle. The leftmost A brane can be brought past the $SO(8)$ stack with an orientation reversal, so that the order becomes $AAAABCAC$. This structure will be important in what follows, because the movement in moduli space that we will use to study instanton zero modes involves "popping off" the AC associated to each E_6 singularity, leaving behind an $SO(8)$ singularity. It is also important to note that

$$\mathcal{M}_{AC} = \mathcal{C} \mathcal{A} = \begin{pmatrix} 2 & -3 \\ 1 & -1 \end{pmatrix}, \tag{3.9}$$

from which it can be seen that AC also leaves $\tau = e^{\frac{\pi i}{3}} + 1$ invariant.

In addition to acting on the axio-dilaton τ, the monodromy associated with a generic 7-brane also acts on the field strengths $H_3 \equiv dB_2$ and $F_3 \equiv dC_2$ of the Neveu-Schwarz B-field and the Ramond-Ramond two-form. This means, for example, that by crossing the branch cut associated with a generic 7-brane, the B_2 charge associated with the fundamental string becomes a combination of NS and RR charge, so that the fundamental string becomes a string with both F-string and D-string charge. A string with p units of F-string charge and q units of D-string charge is known as a (p, q)-string, which are the types of strings that can end on (p, q) 7-brane.

It is known [64] that a string which crosses the branch cut of a (p, q) 7-brane is equivalent to a configuration where the string has passed through the brane and grown a prong by a process that is the U-dual of the Hanany-Witten effect [65]. The resulting configuration is a string junction of the kind that we have encountered in the previous section. Unlike fundamental strings, which only can give two-index representations of the classical Lie algebras, string junctions can also fill out representations of exceptional algebras [61]. We have depicted the equivalence between crossing a branch cut and growing a prong, together with the type and multiplicity of prongs in the junction, in figure 3. Junctions between a $D3$ brane probe and a set of 7 branes are also known to reproduce the BPS states of $\mathcal{N} = 2$ $d = 4$ theories with ADE flavor symmetry [66], where the 7 branes define the flavor algebra. Such junctions can also be used to study charged instanton zero modes if the junction is connected to a $D3$ brane transverse to the 7-branes, rather than to a $D3$ probe. We will proceed to do so in the rest of this section.

3.2 F-theory on K3: from strongly coupled E_6^3 to $SO(8)^4$ at weak coupling

We present an illustrative example using F-theory on an elliptically fibered $K3$. We choose this example because it is simple, while still allowing for the type of physics we would like to study, since the moduli space of $K3$ contains regions of weak coupling and regions of
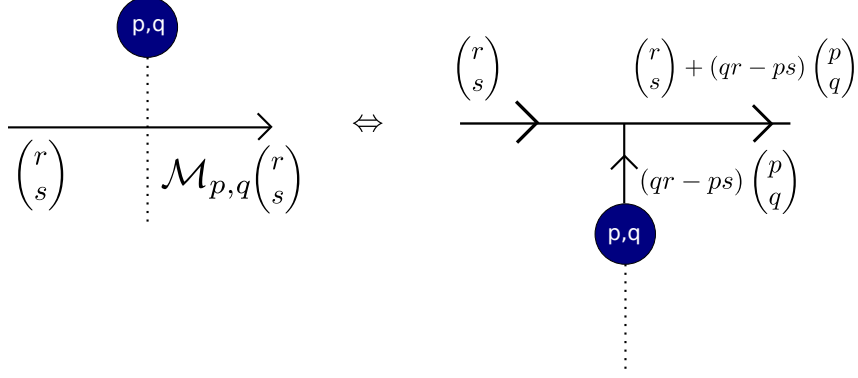
Figure 3. The transformation of an (r, s)-string upon crossing the branch cut associated with a (p, q) 7-brane. On the right is a junction configuration, which is equivalent to the original configuration on the left via the Hanany-Witten effect.

strong coupling with exceptional singularities. In particular, there is a path in the moduli space of $K3$ which interpolates between three E_6 singularities at strong coupling and four $SO(8)$ singularities at weak coupling, which then allows for the appearance of sixteen D7-branes and four O7-planes, as usual. Wrapping a euclidean D3 on the $\mathbb{P}^1$ base of $K3$, we show that some of the standard sixteen 3-7 strings of weakly coupled type IIB (λ modes) are states which contribute to **27**'s of E_6, while others are massive string junctions which become massless in certain regions of moduli space.

Our study of λ modes as string junctions will involve following the movement of the junctions on a particular path in complex structure moduli space. This path includes both strong coupling and weak coupling points, but always maintains *constant* coupling. It is important to note that the constant coupling limit is different from Sen's limit, which is a weak coupling limit with possibly varying (though only slightly) coupling. On some regions of moduli space they will overlap. For example, the region of the moduli space at weak coupling with $SO(8)^4$ gauge symmetry is also at constant coupling, since the D7-branes locally cancel the tadpoles of the O7-planes.

Let us review the constant coupling limit [51] of F-theory on $K3$. We write elliptically fibered $K3$ in Weierstrass form as

$$y^2 = x^3 + fx + g, \tag{3.10}$$

where $f \in H^0(\mathbb{P}^1, K_{\mathbb{P}^1}^{-4})$ and $g \in H^0(\mathbb{P}^1, K_{\mathbb{P}^1}^{-6})$. Since $\mathbb{P}^1$ is $\mathbb{C}$ with a point at infinity, we can go to a patch while still being fairly general, which turns f and g into (generically) inhomogeneous polynomials of degree 8 and 12 in a complex coordinate z. The modular parameter τ of the fiber determines the j-function as

$$j(\tau) = \frac{4(24f)^3}{4f^3 + 27g^2} . \tag{3.11}$$

Let us go to a region of moduli space where $f(z) = \alpha(\phi(z))^2$ and $g(z) = \phi(z)^3$, with $\phi(z) = \prod_{i=1}^4 (z - z_i)$ an arbitrary polynomial of degree four. In this case the j-function

becomes

$$j(\tau) = \frac{55296\alpha^3}{4\alpha^3 + 27}. \tag{3.12}$$

It is clear that the j-function, and therefore τ, is constant on the base, and therefore this region of $K3$ moduli space corresponds to F-theory at constant coupling. If one restricts even further where we are in moduli space by tuning α such that $4\alpha^3 + 27$ vanishes, the j-function blows up, which corresponds to $g_s \to 0$. This is precisely the specialization of Sen's limit of F-theory on K3 to constant coupling. In general, however, the constant coupling parameterization giving (3.12) is *not* at weak coupling.

It was pointed out in [63] that there are other branches of the constant coupling moduli space. One such branch, useful for our purposes, comes about in the limit $\alpha \to 0$, so that f is identically zero and thus $j = 0$. This corresponds to $\tau = e^{\frac{i\pi}{3}} + 1$. This branch of moduli space is fundamentally different than the $\alpha \neq 0$ branch, since $f = 0$ allows g to be an arbitrary polynomial of degree 12 in z while maintaining constant coupling. The two branches meet at $\alpha = 0$ when g is the cube of a quartic polynomial. Out on the $\alpha = 0$ branch, one could write (for example)

$$f = 0 \qquad\qquad g = \prod_{n \in \{0,1,2\}} (z - e^{\frac{2\pi ni}{3}})^3 (z - \beta\, e^{\frac{2\pi ni}{3}}) \tag{3.13}$$

where β is a parameter that can be tuned to move on this branch. The intersection with the $\alpha \neq 0$ branch occurs at $\beta = 0$. The singularity type and position of the branes as one moves from $\beta = 1$ to $\beta = 0$ is depicted in figure 4. Physically, $\beta = 1$ corresponds to a point with three stacks of 7-branes with E_6 gauge symmetry, which can be represented in terms of A, B, and C branes as $AAAAABC^2$, as discussed in section 3.1.
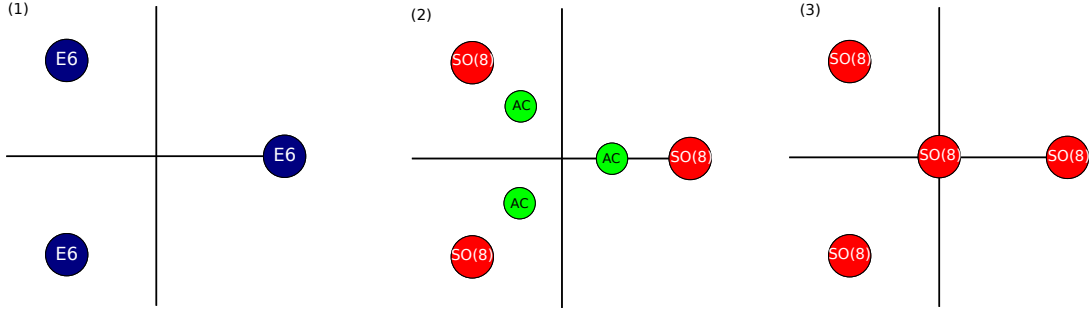
Figure 4. Depicted above is the gauge symmetry and position of 7-branes as one moves from $\beta = 1$ to $\beta = 0$ in the moduli space of $K3$. This $\alpha = 0$ branch of the moduli space has $\tau = e^{\frac{\pi i}{3}}$ and all branes depicted have monodromy such that τ is constant under taking a circle around them.

As one tunes β to be less than one, one A brane and one C brane are pulled off of each E_6 stack, leaving behind three copies of $A^4BC = SO(8)$. When $\beta = 0$, the three sets of AC branes have come together to form an $(AC)^3$ stack, which has the monodromy of an $SO(8)$ (we will present an explicit brane motion that takes a $(AC)^3$ stack to the conventional A^4BC presentation of $SO(8)$ below). At $\beta = 0$, it is important to note that g is the cube of a quartic polynomial, as in the original constant coupling parameterization.

One can then turn on α and tune it to $\alpha = -(\frac{27}{4})^{\frac{1}{3}}$, which is the weak coupling limit.

To summarize, the path in moduli space which we study is as follows. Starting from the E_6^3 point at $\tau = e^{\frac{\pi i}{3}} + 1$, move to the $SO(8)^4$ point at the same τ. At these points $g_s \sim \mathcal{O}(1)$, and we can then send $g_s \mapsto 0$ by sending α from 0 to $-(\frac{27}{4})^{\frac{1}{3}}$ while maintaining $SO(8)^4$ gauge symmetry. We are then in weakly coupled type IIB.

3.3 Exceptional charged modes, massive BPS states, and relating to IIB

To this point, we have demonstrated that there is a path in the moduli space of $K3$ which interpolates between E_6^3 and $SO(8)^4$ at $\tau = e^{\frac{i\pi}{3}} + 1$, and then goes to $SO(8)^4$ at weak coupling. From that region of moduli space, the $D7$-branes and $O7$-planes can spread out over the $\mathbb{P}^1$, at which point it is clear that the 3-7 strings are the standard λ modes of weakly coupled type IIB. What becomes of the λ modes as one moves in moduli space between the $SO(8)^4$ point at weak coupling and the E_6^3 point at $\tau = e^{\frac{i\pi}{3}} + 1$?

Recall that the instanton wraps the $\mathbb{P}^1$ base of $K3$ and two directions of $8d$ spacetime. At generic points in moduli space, there is some set of 7 branes intersecting the instanton at a complex codimension one defect in its worldvolume. Massless charged modes are given by 3-7 junction of zero length.[8] That is, a zero mode represented by a string junction can attach to one set of localized (p, q) 7-branes, but not to multiple sets which are separated in the $\mathbb{P}^1$. Given a set of localized 7-branes, it was shown in [61] how to classify the junctions ending on that set and determine the representation of those junctions.

Of importance is the notion of asymptotic charge, which is nothing but the (p, q) type of a free prong. For example, if the junction in figure 3 had its prong with charge (r, s) attached to an (r, s) 7-brane, then there would be one free end with asymptotic charge $(r + (qr - ps)p, q + (qr - ps)q)$. Since the free ends of the junctions we study will attach to a $D3$ brane and we want to relate those junctions to the standard charged instanton zero modes in the IIB limit, we will focus on junctions with asymptotic charge $(1, 0)$. That is, we will study junctions which are asymptotically fundamental strings.

3.3.1 The fate of junctions in the 27 of E_6

At the E_6^3 point in moduli space, the charged instanton zero modes we study are those of asymptotic charge $(1, 0)$.[9] To remain massless, the junctions cannot stretch between E_6 stacks, and thus the study of zero modes at the E_6^3 point amounts to the study of string junctions of asymptotic charge $(1, 0)$ connecting to a single E_6 stack, picking up a multiplicity of 3 for the three stacks.

The set of junctions of asymptotic charge $(1, 0)$ and self-intersection -1 attached to an E_6 stack represented by left-right ordering $AAAAABCC$ were classified in [61]. There

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[8]We will use the terminology "zero modes" for zero length junctions. In view of the discussion in section 2, we should more properly be talking about massless carriers, since there we saw that the map between junctions and zero modes is subtle. We have not performed the analogous anomaly inflow analysis for charged modes, so we will stick to this slightly imprecise nomenclature. We hope to come back to this question in future work.

[9]There is a further condition on the junction: it should have self-intersection equal to 1 [61]. The states that we study below all have this property.

are four types of such junctions, given by

$$
\begin{aligned}
\text{Type I:} \quad & a_i \\
\text{Type II:} \quad & -a_i + b + c_j \\
\text{Type III:} \quad & -\sum_{p=1}^{3} a_{i_p} + 2b + c_1 + c_2 \\
\text{Type IV:} \quad & -\sum_{p=1}^{5} a_p + 3b + c_k + c_1 + c_2,
\end{aligned}
\tag{3.14}
$$

where the appearance of a term a_i denotes that the junction has a prong on the ith A brane, with a $+$ sign denoting outgoing and a $-$ sign denoting incoming, with the terms c_i similarly defined. The A brane indices run from $1 \ldots 5$ and the C brane indices from $1 \ldots 2$, as one would expect. Junctions of Type III end on three of the five A branes, with $i_1 < i_2 < i_3$. A simple counting shows that there are 5 junctions of Type I, 10 junctions of Type II, 10 junctions of Type III, and 2 junctions of Type IV, for a total of 27 junctions. Moreover, they have the Lie algebraic structure of a **27** of E_6.

What happens to these junctions as one moves in moduli space? Recall that the path we have defined interpolating between E_6^3 and $SO(8)^4$ involves removing an A brane and a C brane from each E_6, leaving an $SO(8)$ behind. Any junction in the **27** with a prong on the removed A or C brane becomes of finite length upon moving in the defined path on moduli space. Thus, all junctions of Type III and Type IV have finite length. Six junctions of Type II have finite length while the other four are left behind, localized at the $SO(8)$. All junctions of Type I remain of zero length, though one is localized on the A brane that has been removed, while the other four are localized at the $SO(8)$. This gives a total of 8 junctions that are still of zero length and localized at the $SO(8)$. These junctions, depicted in figure 5 are known to fill out an $\mathbf{8}_v$ of $SO(8)$.

In summary, of the 27 junctions filling out a **27** of E_6, 18 become massive states, 1 is a massless junction attached to the removed A brane, and 8 are left behind as massless junctions that fill out an $\mathbf{8}_v$ of $SO(8)$. Since there are three E_6 stacks initially, there is an overall multiplicity of three for all of the mentioned junctions. This leaves a puzzle, however: in type IIB at the $SO(8)^4$ point, there should be four sets of charged instanton zero modes transforming in the $\mathbf{8}_v$ of $SO(8)$, whereas in the present analysis we have seen that only three copies of $\mathbf{8}_v$ arise from Higgsing **27**'s of E_6. More specifically, there are 3 massless junctions at the fourth $SO(8)$ coming from the Type I junctions associated with the removed A brane of each E_6, but there are still 5 junctions in the fourth $\mathbf{8}_v$ which are not accounted for.

3.3.2 The fourth $\mathbf{8}_v$ of $SO(8)$ and massive states

There is one complication that we must discuss before addressing the appearance of the fourth $\mathbf{8}_v$ as a continuous process upon movement in moduli space. The junctions that fill out an $\mathbf{8}_v$ are given in figure 5, but those junctions are for the case where the $SO(8)$ is realized as $AAAABC$. In the case of moving from E_6^3 to $SO(8)^4$, the fourth $SO(8)$ stack
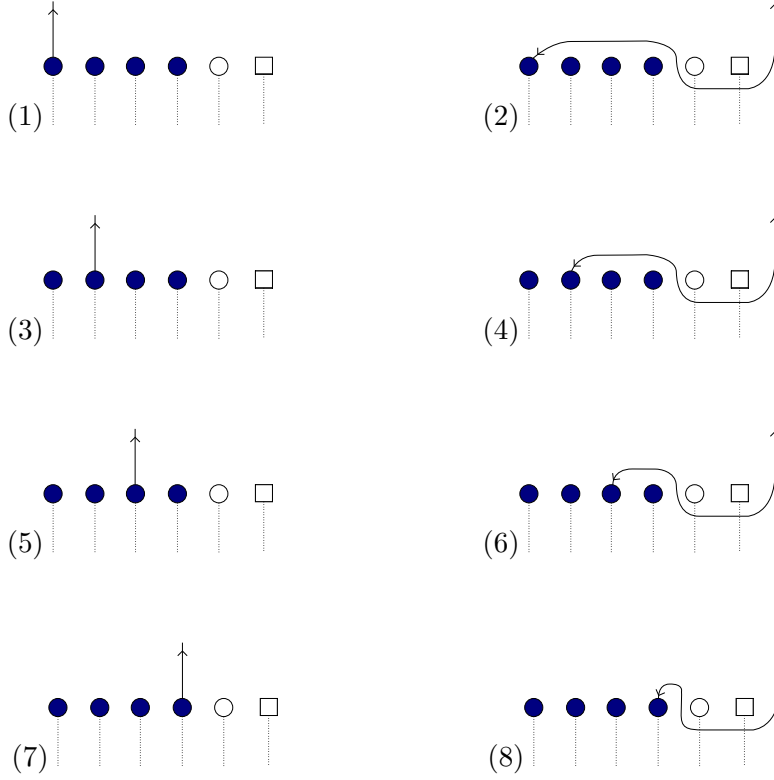
Figure 5. Upon tuning $\beta < 1$, an A and a C brane are pulled away from the E_6 stack, leaving an $SO(8)$ stack. Depicted are the 8 junctions out of the **27** which are localized at the $SO(8)$ after tuning $\beta < 1$. These junctions fill out an $\mathbf{8}_v$ of $SO(8)$.

located at $z = 0$ has left-right ordering $ACACAC$. Before we can discuss the appearance of the $\mathbf{8}_v$ we must perform a technical exercise to determine how the junctions forming an $\mathbf{8}_v$ of $SO(8)$ as $AAAABC$ are transformed under the brane movement $AAAABC \mapsto ACACAC$. In appendix B we explicitly untangle each junction that contributes to the $\mathbf{8}_v$ of $AAAABC$ so that we can discuss what becomes of the fourth $\mathbf{8}_v$ upon movement in moduli space. Here we present one of the more difficult examples in detail, so that our method is clear.

Consider the example of junction (3) of figure 9 in appendix B, which we will show becomes junction (4) after untangling. This involves determining a way to take the branes around the nearby branch cuts to transform $AAAABC$ into $ACACAC$. It also requires keeping track of the string as one moves the branes. The starting configuration is junction (3)
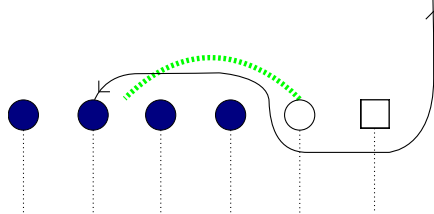
where we have represented A-branes, B-branes, and C-branes as filled circles, hollow circles,

and hollow squares, respectively. The (p,q) string is represented by a solid line, and the arrow into the A-brane indicates a $(1,0)$ string going in, or equivalently a $(-1,0)$ string coming out. It has asymptotic charge $(1,0)$. The dotted green line represents the path taken for the next step in the untangling process. Taking that path transforms the configuration to
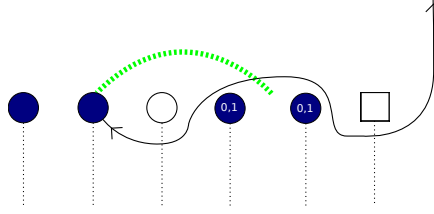
where branes labeled with white numbers (p,q) denote a brane of that type, and the green path again denotes the next brane movement in the untangling process. The two branes are $(0,1)$ type because they have passed the branch cut of the B brane that was moved. Taking the next movement, we arrive at
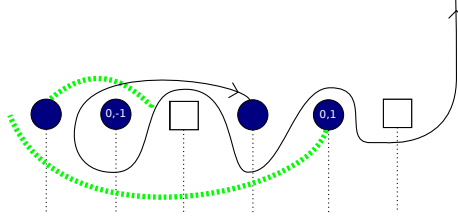
and then takes the final two movements to arrive at
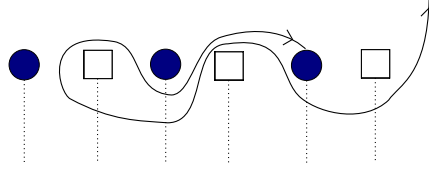
As a double check, if one starts with the $(-1,0)$ string attached to the rightmost A brane and applies the appropriate $SL(2,\mathbb{Z})$ transforms, one sees that this configuration does indeed have asymptotic charge $(1,0)$, as it must.

In untangling the branes and strings, here and in appendix B, we have not used the Hanany-Witten effect to change strings crossing branch cuts into junctions. The main reason for this is that fewer steps offer more simplicity, and we have found that it also makes comparing states a bit easier. Nevertheless, as an example let us convert the final (p,q) string configuration just derived into a junction. Applying the rules for converting

(p, q) strings crossing branch cuts into junctions, as exemplified in figure 3, one arrives at

which looks quite complicated. This junction is massless when the three AC combinations from the E_6's coalesce at $\beta = 0$ and contributes to an $\mathbf{8}_v$. If one considering the opposite path in moduli space, though, moving from $\beta = 0$ to $\beta = 1$, the junction is such that there will be prongs stretching between different E_6 stacks. That is, this component of the fourth $\mathbf{8}_v$ is a massive BPS states at the E_6^3 point in moduli space.

In fact, upon moving to the E_6^3 point, five of the junctions in the fourth $\mathbf{8}_v$ stretch between distinct E_6 stacks and are therefore massive. Junctions (2) and (4) of figure 8 and junction (8) of figure 9 are the three which do not become massive upon moving to the E_6^3 point. They are precisely the three junctions of Type I that contribute to the $\mathbf{27}$'s, one for each E_6, that were discussed in section 3.3.1.

3.4   A brief summary and outlook

We have seen that string junctions are useful for studying 3-7 instanton zero modes in F-theory, even upon movement in moduli space. We defined a very specific path in the complex structure moduli space of F-theory on $K3$ which interpolated between E_6^3 symmetry at strong, constant coupling and $SO(8)^4$ symmetry in weakly coupled type IIB.

At the E_6^3 point in moduli space, the 3-7 instanton zero modes with charge (1,0) ending on the D3 instanton are three $\mathbf{27}$'s of E_6. In terms of A branes, B branes, and C branes E_6 is represented as $AAAAABCC$. Moving from the E_6^3 point to the $SO(8)^4$ point, an A brane and a C brane are pulled off of each E_6 stack, which coalesce to form the fourth $SO(8)$. Of the 27 junctions filling out each $\mathbf{27}$ at the E_6^3 point, 18 become massive BPS states with prongs between separated branes, 1 is a fundamental string attached to the A brane which was removed, and 8 form the $\mathbf{8}_v$ of the $SO(8)$ that was left behind when the AC combination was removed. Those 3 fundamental strings associated with each A brane that was removed are three of the junctions necessary to form the fourth $\mathbf{8}_v$ which is known to be present in type IIB. The other 5 junctions contributing to the fourth $\mathbf{8}_v$ are massive BPS states for $\beta > 0$ that become massless when $\beta = 0$.

The appearance of additional light states from ones that are massive at generic points in moduli space is common in string theory. In F-theory it is simple to see in the junction picture, because junctions between separated branes have finite length, but become massless when the branes come together. The statements we have made about massless and

massive junctions upon movement in moduli space are not limited to charged instanton zero modes, though. In particular, though we were interested in attaching the "free end" with asymptotic charge $(1,0)$ to a $D3$ instanton transverse to the 7-branes, it could also attach to an A brane, or a $D3$ parallel to the 7-branes. In that case, one would have extra matter, rather than extra instanton zero modes, as one moves in moduli space. In realistic scenarios involving instantons one will have a combination of both phenomena at play, so we now proceed to study situations in which both light matter and extra zero modes appear at particular points in moduli space.

4 Instanton physics via heterotic / F-theory duality

In section 3 we demonstrated that string junctions provide a useful tool for uncovering interesting relationships between charged instanton zero modes in F-theory and type IIB. In an example, we showed that to reproduce known type IIB results smoothly from F-theory one has to take into account both the Higgsing of massless junctions and massive junctions which become massless at certain points in moduli space. We expect this behavior to be fairly common. Though the string junction picture gives a consistent and illuminating picture of charged zero modes, it is still not clear how to use such knowledge to actually compute superpotential corrections. In this section we utilize duality between the heterotic string and F-theory to compute (via duality) instanton corrections in F-theory at points in moduli space with exceptional gauge symmetry. On the heterotic side, worldsheet instantons provide the superpotential corrections we will study [67–71]. Of particular interest to us is that [42, 43] obtained the dependence of the worldsheet instanton generated superpotential on the heterotic vector bundle moduli space. This dualizes to ED3/M5 instanton corrections in F-theory depending on the fourfold complex structure moduli. Microscopically, the means that the structure of the correction is dependent on the position of 7-branes, due to the appearance of extra zero modes when the positions and structure of 7-branes take a particular form. By reinterpreting the known answer from the heterotic in the F-theory language we will obtain information about the physics away from weakly coupled limits.

For the convenience of the reader, since we introduce a good deal of notation in what follows, we collect our naming and notational conventions here. As will be discussed in section 4.1, heterotic / F-theory duality requires a number of fibration structures, given by

$$
\begin{aligned}
T^2 &\longhookrightarrow X \xrightarrow{\ \pi_H\ } B_2 & T^2 &\longhookrightarrow Y \xrightarrow{\ \pi_F\ } B_3 \\
K3 &\longhookrightarrow Y \xrightarrow{\ \pi_{K3}\ } B_2 & \mathbb{P}^1 &\longhookrightarrow B_3 \xrightarrow{\ \pi_{\mathbb{P}^1}\ } B_2,
\end{aligned}
\tag{4.1}
$$

where X is the heterotic elliptic CY_3 and Y is the F-theory elliptic CY_4 that is also $K3$ fibered. Our convention is to name projection maps with a subscript denoting the fiber. The only ambiguity is for the two elliptic fibrations, in which case we name the maps π_H and π_F for heterotic and F-theory, respectively. We give a summary of the notation used in this section in tables 1 and 2.

Symbol	Definition and/or Comment
X	Elliptic CY_3 of the heterotic compactification.
σ_H	Section of X.
Y	Elliptic CY_4 of F-theory. Also $K3$ fibered.
σ_F	Section of Y.
B_2	Twofold base of X and Y.
B_3	Threefold base of Y. It is a $\mathbb{P}^1$ fibration over B_2.
Σ	Curve in B_2 wrapped by the instanton.
$\mathcal{E}$	$\pi_H^{-1}(\Sigma)$.
$\pi_{K3}^{-1}(\Sigma)$	Divisor wrapped by $M5$ instanton.
χ	$\Sigma \cdot c_1(TB_2)$

Table 1. Table of notation for geometric objects used in discussing heterotic / F-theory duality and instantons. Please see the text for context and discussion.

| Symbol	Definition and/or Comment
V	Holomorphic vector bundle on X, $V = V_1 \oplus V_2$.
V_1	Bundle studied for WS instanton corrections.
(C, L)	Spectral pair equivalent to V_1 under Fourier-Mukai.
C	A divisor in X of class $n\,\sigma_H + \pi_H^{-1}\eta$, with η a curve in B_2
f_C	Polynomial whose zero locus is C.
a_q	Sections of line bundles on B_2 appearing in f_C.
c	$c \equiv C \cdot \mathcal{E}$. Divisor in $\mathcal{E}$ of class $n\sigma_{\mathcal{E}} + rF$
f_c	Polynomial whose zero locus in $\mathcal{E}$ is c. $f_c \equiv f_C|_{\mathcal{E}}$
$\tilde{a}_q$	Section of line bundles on Σ appearing in f_c.
L	Line bundle on X, gives a line bundle on C via restriction.
$\tilde{\mathcal{L}}$	Line bundle on $\mathcal{E}$, $\tilde{\mathcal{L}} \equiv L|_{\mathcal{E}} \otimes O_{\mathcal{E}}(-F)$.
$\mathcal{L}$	Line bundle on c, $\tilde{\mathcal{L}} \equiv \tilde{\mathcal{L}}|_c$. $pfaff_{\Sigma} = 0 \Leftrightarrow h^0(c, \mathcal{L}) > 0$.

Table 2. Table of notation for objects related to the rank n holomorphic vector bundle V on X.

4.1 Heterotic / F-theory duality

We begin by reviewing the basics of heterotic / F-theory duality, focusing on the aspects most relevant for our discussion. For a more detailed summary with an emphasis on F-theory GUTs, see for example [72]. The simplest example is that the $E_8 \times E_8$ heterotic string on T^2 is dual to F-theory on $K3$. The heterotic side is endowed with a rank n holomorphic vector bundle over T^2 which encodes the breaking of $E_8 \times E_8$. On the F-theory side, the bundle moduli which specify the symmetry breaking are encoded purely in terms of geometry. That is, F-theory gives a geometric depiction of the moduli space of holomorphic vector bundles on T^2. By making this precise, we will be able to map results from heterotic compactifications to F-theory.

In six dimensions, the heterotic string on $K3$ is dual to F-theory compactified on a Calabi-Yau threefold which is an elliptic fibration over a Hirzebruch surface $\mathbb{F}_p$ [73, 74]. This can be understood by fibering the compactification manifolds in the eight-dimensional case over a $\mathbb{P}^1$. Since the Hirzebruch surface is a $\mathbb{P}^1$ fibration over $\mathbb{P}^1$, the F-theory CY_3 is also a $K3$-fibration over $\mathbb{P}^1$. That is, heterotic on an elliptic Calabi-Yau over B is dual to
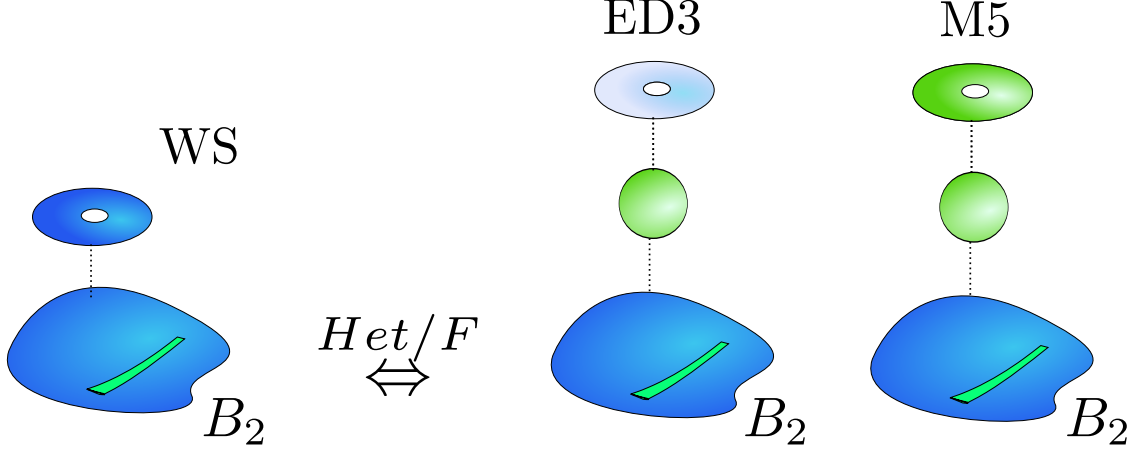
Figure 6. A depiction of a heterotic worldsheet instanton wrapped on a curve Σ in B_2 and the dual $ED3/M5$ instanton in F-theory. The heterotic T^2 becomes a $\mathbb{P}^1$ base of the $K3$ fiber in F-theory which is wrapped by the instanton. Green denotes the instanton wrapping that part of the geometry. On the right we show both the IIB and M-theory descriptions of the dual F-theory. In the first the elliptic fiber is an auxiliary description for the variation of the axio-dilaton τ, and the physical object is an euclidean $D3$, while in the second the fiber is physical, and the $M5$ brane instanton wraps it.

F-theory on a $K3$-fibered Calabi-Yau over B, with here $B = \mathbb{P}^1$.

As one might expect, a similar story holds in four dimensions. In this paper, we will focus on cases where the heterotic string is compactified to four dimensions on an elliptic Calabi-Yau X with section σ_H over a two-fold base B_2, which is dual to F-theory on a $K3$-fibration over B_2. We take homogeneous fiber coordinates $(x_H, y_H, z_H) \in \mathbb{P}_{231}$ so that the X is given by the Weierstrass equation

$$y_H^2 = x_H^3 + \tilde{f}\, x_H z_H^2 + \tilde{g}\, z_H^6, \tag{4.2}$$

where $\tilde{f}$ and $\tilde{g}$ are sections of $K_{B_2}^{-4}$ and $K_{B_2}^{-6}$, respectively. The F-theory compactification manifold is an elliptic Calabi-Yau over B_3, which is itself a $\mathbb{P}^1$ fibration over B_2, and has Weierstrass equation

$$y_F^2 = x_F^3 + f\, x_F z_F^2 + g\, z_F^6, \tag{4.3}$$

where f and g are sections of $K_{B_3}^{-4}$ and $K_{B_3}^{-6}$. When the need to specify arises, we will take $B_2 = \mathbb{F}_p$, so that B_3 is a generalized Hirzebruch variety $\mathbb{F}_{pmk}$. (A generalized Hirzebruch variety is a $\mathbb{P}^1$ fibration over a Hirzebruch surface. We include some background material on this geometry in appendix A.)

Heterotic / F-theory duality is made manifest by using the spectral cover formalism [75, 76], in which a holomorphic vector bundle V is determined by specifying a spectral surface C and a holomorphic line bundle L on it. The pair (C, L) determines V by Fourier-Mukai transform. V can be decomposed as $V = V_1 \oplus V_2$, with V_1 associated to one E_8 factor and V_2 to the other. The geometric moduli which determine C map directly to complex structure moduli in F-theory. For concreteness and since it is what we need to

dualize the worldsheet instanton calculation, we focus on a single bundle V_1.

As a divisor in X, the spectral surface C has homology class

$$C = n\sigma_H + \pi_H^* \eta \qquad (4.4)$$

for some curve η in B_2 which is determined by $c_2(V_1)$ under Fourier-Mukai transform. Deformation moduli of the spectral surface C are bundle moduli which are counted by $h^0(X, \mathcal{O}_X(C))$. One can decompose sections of $h^0(X, \mathcal{O}_X(C))$ appearing in the polynomial f_C, whose zero locus determines C, in terms of fiber coordinates and sections of the base as

$$f_C = a_0 \, z_H^n + a_2 \, x_H z_H^{n-2} + a_3 \, y_H z_H^{n-3} + a_4 \, x_H^2 z_H^{n-4} + a_5 \, x_H y_H z_H^{n-5} + \dots, \qquad (4.5)$$

where a generic term $a_q x_H^{n_x} y_H^{n_y} z_H^{n-q}$ (subject the constraint $q = 2n_x + 3n_y$) has

$$a_q \in H^0(B_2, K_{B_2}^{\otimes q} \otimes \mathcal{O}_{B_2}(\eta)) \cong H^0(B_2, K_{B_2}^{\otimes q-6} \otimes \mathcal{O}_{B_2}(\tilde{\eta})). \qquad (4.6)$$

As will become clear, it is convenient to write $\eta \equiv \tilde{\eta} - 6[K_{B_2}]$. Intuitively, one might expect that as the rank n of V_1 increases, the additional monomials allowed in (4.13) correspond to degrees of freedom that further break the E_8 factor. That is, a_q degrees of freedom for increasingly large q correspond to larger amounts of breaking. This fact can be seen plainly on the heterotic side, but we will present it from the F-theory point of view, where the spectral surface moduli[10] control the degenerations of the elliptic fiber, and hence the structure of 7-branes.

The spectral line bundle L on C is often presented as the restriction of a line bundle $\underline{L}$ on X. To avoid introducing too much notation, we will abuse notation and refer to $\underline{L}$ simply as L, as is common in the literature, so that L is a line bundle on X which restricted to C gives the spectral line bundle, also called L. We will mostly deal with bundles of $SU(3)$ structure. The constraint $c_1(V) = 0$ then requires that

$$c_1(L) = \frac{n}{2}\sigma_H + \frac{1}{2}\pi_H^* \eta + \frac{1}{2}\pi_H^* c_1(TB_2) + \gamma, \qquad (4.7)$$

so that L is determined up to a choice of a twist parameter γ, of the form

$$\gamma = \lambda(n\sigma_H - (\pi_H^* \eta - nc_1(TB_2))), \qquad (4.8)$$

with λ integral if n is even and λ half integral if n is odd. This parameter maps into G-flux in F-theory, where the parameter λ controls the "amount" of flux and the choice of η determines its structure.

4.1.1 Mapping bundle moduli

We now wish to identify complex structure moduli on the F-theory side with vector bundle moduli on the heterotic side. The basics of the discussion are presented in [77], and we

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[10]For the time being, we only consider those bundle moduli which correspond to spectral surface moduli.

fill in some details relevant for our instanton computation, where the bundle moduli are restricted to the curve in the base which the worldsheet instanton wraps.

As discussed, the compactification manifold on the F-theory side is a Calabi-Yau four-fold Y that is an elliptic fibration over B_3, which itself is a $\mathbb{P}^1$ fibration over B_2. Y is also a $K3$ fibration over B_2. The dual seven-branes wrap the base B_2 of the $K3$-fibration. We decompose f and g, which are sections of line bundles on B_3, in terms of the homogeneous coordinates y and z of the $\mathbb{P}^1$ fiber of B_3 and sections of B_2. This gives

$$f = \sum_{a=0}^{8} y^{8-a} z^a \ f_a \qquad\qquad g = \sum_{b=0}^{12} y^{12-b} z^b \ g_b, \qquad (4.9)$$

where the sections f_a and g_b are

$$
\begin{aligned}
f_a &\in H^0(B_2, K_{B_2}^{-4} \otimes \mathcal{O}_{B_2}(\tilde{\eta})^{\otimes(4-a)}) \\
g_b &\in H^0(B_2, K_{B_2}^{-6} \otimes \mathcal{O}_{B_2}(\tilde{\eta})^{\otimes(6-b)}).
\end{aligned}
\qquad (4.10)
$$

From this parameterization, it is clear that in much of the moduli space the discriminant $\Delta = 4 f^3 + 27 g^2$ has zeros at $z = 0$, signaling the presence of a 7-brane at $z = 0$ which wraps the base B_2 of the $K3$ fibration. In this notation, f_4 and g_6 are equivalent to $\tilde{f}$ and $\tilde{g}$, which determine the Weierstrass equation for the heterotic elliptic Calabi-Yau X. From (4.10) it is easy to see that they do not depend on $\tilde{\eta}$, which is data associated to the heterotic bundle V. Rather, they depend only on geometry of the heterotic compactification.

For our purposes, we will often (although not exclusively) focus on the gauge group along the 7-brane at $z = 0$ at various points in complex structure moduli space. It is known from the Kodaira classification that an E_8 singularity has orders of vanishing $ord(f) \geq 4$, $ord(g) = 5$, $ord(\Delta) = 10$. Thus one can take $f = 0$ [11] and $g_0 = g_1 = g_2 = g_3 = g_4 = 0$. If g_5 does not vanish at a point in moduli space, then g vanishes to order 5 and Δ vanishes to order 10 at $z = 0$, ensuring an E_8 singularity there. That is, the moduli specifying g_5 are the moduli which preserve an E_8 singularity.

It is known on the heterotic side that the moduli specifying a_0 preserve one E_8 if the higher a_q's vanish. In [77] a careful counting showed that a_0 and g_5 have the same number of moduli. The simple reason for this, as seen from (4.6) and (4.10) is that g_5 and a_0, are sections of the same bundle, and thus they should be identified. One statement of heterotic / F-theory duality, then, is that the mathematical object a_0 which specifies part of a vector bundle on the heterotic side contributes purely to the geometry of a $K3$ on the F-theory side. By similar arguments, one can show that moduli in f_3 preserve an E_7 singularity, which is identified with a_2 since they are sections of the same bundle. The section that preserves an E_6 is g_4. From an initial glance at (4.6), it is easy to see that there is no single a_q which can be identified with g_4, due to the tensor power of the $\mathcal{O}$-bundle in g_4, but that a_3^2 and g_4 are sections of the same bundle. One might be concerned that g_4 should

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[11]This might seem strange, but recall that in the case of $K3$ in section 3 we took $f = 0$ to get an E_6 singularity, which has $ord(f) \geq 3$, $ord(f) = 4$, $ord(\Delta) = 10$. That is, f can vanish everywhere, as long as g, Δ vanish to the correct order.

not be identified with a_3^2, since g_4 in its most general form does not have to be a perfect square. If g_4 were not a perfect square, though, the singular fiber would undergo non-trivial monodromy upon taking certain paths in B_2 and the singularity structure would be F_4 rather than E_6 [77, 78]. Thus, we have $g_4 \equiv a_3^2$ if E_6 is to be preserved.

4.2 Heterotic worldsheet instantons and their F-theory duals

Having reviewed the basics of heterotic F-theory duality necessary for our discussion of instantons, we now discuss heterotic worldsheet instantons and their ED3/M5 instanton duals in F-theory.

Non-perturbative worldsheet instanton effects in heterotic string vacua [67–70] can give corrections to the superpotential that play important roles in low energy physics, lifting vector bundle moduli (see [79–81] for early examples) or giving corrections [71] to couplings involving charged matter fields. Consider the compactification of the heterotic $E_8 \times E_8$ theory on a smooth Calabi-Yau threefold X endowed with a holomorphic vector bundle V. Then a superpotential correction due to a heterotic worldsheet instanton wrapped on a holomorphic curve Σ in X generically takes the form

$$W_\Sigma \;\sim\; \mathit{pfaff}_\Sigma \; e^{i \int_\Sigma J} \tag{4.11}$$

where pfaff_Σ is the one-loop Pfaffian prefactor and J is the complexified Kähler form. The Pfaffian prefactor is a function of vector bundle moduli and greatly influences the structure of the superpotential correction, including its zeroes. This will be the main object of our study via duality. It was computed directly using algebro-geometric techniques in [42, 43], where in the examples studied it took the form

$$\mathit{pfaff}_\Sigma \;\sim\; p^k q \qquad\qquad k \in \mathbb{Z}, \tag{4.12}$$

with p and q being complicated polynomials in the vector bundle moduli which define the spectral surface C. We will show that the structure of p, q, and k have interesting physics in terms of intersecting branes in F-theory. We review the details of the calculation of pfaff_Σ since it is relevant for discussion of the physics of ED3/M5 instantons via duality.

An important condition for the study of worldsheet instanton corrections is that pfaff_Σ (and therefore the superpotential W_Σ) vanishes if and only if the sheaf cohomology $H^0(\Sigma, V_1|_\Sigma \otimes \mathcal{O}_\Sigma(-1))$ is non-trivial[12]. Since we are interested in heterotic compactifications with F-duals, X is elliptically fibered and therefore there must be an isomorphism between $H^0(\Sigma, V_1|_\Sigma \otimes \mathcal{O}_\Sigma(-1))$ and some sheaf cohomology involving only spectral data. Before stating the isomorphism, let us make a few definitions.

Taking π_H the projection of X onto B_2, we define $\mathcal{E} \equiv \pi_H^{-1}(\Sigma)$, $r \equiv \pi_H^* \eta \cdot \mathcal{E}$ and $c \equiv C|_\mathcal{E} = n\sigma_\mathcal{E} + rF$, where r is an integer calculable in specific examples. The divisor c

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[12]For brevity we omit the derivation of this condition, instead referring the reader to [42].

in $\mathcal{E}$ has a defining equation given by[13]

$$f_c = \tilde{a}_0 \ z_H^n + \tilde{a}_2 \ x_H z_H^{n-2} + \tilde{a}_3 \ y_H z_H^{n-3} + \tilde{a}_4 \ x_H^2 z_H^{n-4} + \tilde{a}_5 \ x_H y_H z_H^{n-5} + \dots, \qquad (4.13)$$

where $\tilde{a}_q = a_q|_\Sigma$. It was shown in [42] that

$$H^0(\Sigma, V_1|_\Sigma \otimes \mathcal{O}_\Sigma(-1)) \cong H^0(c, (L|_\mathcal{E} \otimes O_\mathcal{E}(-F))|_c) \qquad (4.14)$$

where F is the elliptic fiber class and a straightforward computation from equation (4.7) gives $L|_\mathcal{E} = \mathcal{O}_\mathcal{E}(n(\frac{1}{2}+\lambda) \ \sigma|_\mathcal{E} + (r(\frac{1}{2}-\lambda)+\chi(\frac{1}{2}+n\lambda)) \ F)$. Here we have defined $\chi = c_1(TB_2) \cdot \Sigma$. This isomorphism gives the sheaf cohomology to be calculated when using the spectral cover formalism and implies that

$$pfaff_\Sigma = 0 \qquad \Leftrightarrow \qquad h^0(c, \mathcal{L}) > 0, \qquad (4.15)$$

where we have defined $\tilde{\mathcal{L}} \equiv L|_\mathcal{E} \otimes O_\mathcal{E}(-F)$ and $\mathcal{L} \equiv \tilde{\mathcal{L}}|_c$ for notational simplicity in what follows. Determining the zeroes of the Pfaffian then amounts to calculating sheaf cohomology. One way of doing this is starting from the short exact sequence

$$0 \to \tilde{\mathcal{L}} \otimes \mathcal{O}_\mathcal{E}(-c) \xrightarrow{f_c} \tilde{\mathcal{L}} \to \mathcal{L} \to 0 \qquad (4.16)$$

and calculating $h^0(c, \mathcal{L})$ via the corresponding long exact sequence in cohomology.

The non-trivial physics is determined by the fact that $h^0(c, \mathcal{L})$ can jump by integral values as one moves in the vector bundle moduli space $\mathcal{M}_X(V)$, signaling the presence of extra zero modes. This was first done in [42], where the map f_c determines matrices $f_c^{(0)}$ and $f_c^{(1)}$ appearing in the long exact sequence

$$0 \to H^0(\mathcal{E}, \tilde{\mathcal{L}} \otimes \mathcal{O}_\mathcal{E}(-c)) \xrightarrow{f_c^{(0)}} H^0(\mathcal{E}, \tilde{\mathcal{L}}) \to H^0(c, \mathcal{L}) \to H^1(\mathcal{E}, \tilde{\mathcal{L}} \otimes \mathcal{O}_\mathcal{E}(-c))$$
$$\xrightarrow{f_c^{(1)}} H^1(\mathcal{E}, \tilde{\mathcal{L}}) \to \dots \qquad (4.17)$$

which gives $h^0(c, \mathcal{L}) = h^0(\mathcal{E}, \tilde{\mathcal{L}}) + h^1(\mathcal{E}, \tilde{\mathcal{L}} \otimes \mathcal{O}_\mathcal{E}(-c)) - rk(f_c^{(0)}) - rk(f_c^{(1)})$. In most cases studied in the literature, one of the groups in (4.17) vanishes so that one must only compute the rank of one of the maps. Jumps in h^0 occur when one of the maps loses rank, which was determined in [42] by calculating the determinants $det(f_c^{(0)})$ and $det(f_c^{(1)})$. These determinants are polynomial functions of moduli appearing in $\tilde{a}_q$ and thus their zero loci determine algebraic subvarieties in the moduli space of deformations of c in $\mathcal{E}$, henceforth called $\mathcal{M}_\mathcal{E}(c)$. The Pfaffian prefactor, and hence the superpotential, vanishes on these subvarieties of moduli space, which we call $(pfaff_\Sigma)$. Our goal is to understand the physics of $(pfaff_\Sigma)$ and further subvarieties in it. In particular, we wish to relate those subvarieties to new features in the low energy 4d theory.

It is illuminating to consider the physics of such a calculation in the dual F-theory

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[13]Here there are two departures from the notation of [82], where we call the defining equation f_c instead of s and the sections $\tilde{a}_q$ instead of a_q, as the a_q's are typically reserved for sections appearing in f_C, not f_c.

compactification to four dimensions. A worldsheet instanton wrapping a curve Σ in X is dual to a vertical $M5$ instanton wrapping $\pi_{K3}^{-1}(\Sigma)$, or alternatively a euclidean D3 wrapping $\mathcal{E}_F \equiv \pi_{K3}^{-1}(\Sigma)|_{B_3} = \pi_{\mathbb{P}^1}^{-1}(\Sigma)$, where we have made use of the multiple fibration structures $Y \xrightarrow{\pi_{K3}} B$ and $B_3 \xrightarrow{\pi_{\mathbb{P}^1}} B$. The two-folds $\mathcal{E}$ and $\mathcal{E}_F$ are related via duality, where the elliptic fiber is mapped to a $\mathbb{P}^1$ as usual.

The F-theory dual of the heterotic zero mode calculation makes physical sense from an open string point of view. The subvariety $(pfaff_\Sigma)$ in $\mathcal{M}_\mathcal{E}(c)$ is a polynomial in moduli of the $\tilde{a}_q$'s. In F-theory these determine the structure of the discriminant Δ over the instanton worldvolume $\mathcal{E}_F$, and therefore $(pfaff_\Sigma)$ is also a subvariety in $\mathcal{M}_{\mathcal{E}_F}(\Delta|_{\mathcal{E}_F})$. Physically, this means that the structure of 7-branes inside the instanton worldvolume (roughly, the structure of 3-7 zero modes) governs whether or not the superpotential vanishes, as is expected from the type II perspective. For this reason, in [34] the curve c was suggestively called Σ_{37}, since the moduli determining c determine the locus of 7-brane intersections with the 3-brane instanton, $\Delta|_{\mathcal{E}_F}$. The Pfaffians calculated on the heterotic side therefore map directly to the F-theory side, where each point in $\mathcal{M}_\mathcal{E}(c)$ determines a brane configuration in the worldvolume of the instanton which governs (along with G-flux) whether or not the superpotential vanishes.[14]

It is worth noting that $\Delta|_{\mathcal{E}_F}$ is determined by the restriction of f_a and g_b to the instanton, which in turn are sections of line bundles determined by the parameter $r \equiv \pi_H^{-1}(\Sigma) \cdot \mathcal{E}$. There are typically multiple curves η which could give rise to the same r, each of which would determine a different heterotic compactification, and therefore a different F-dual. Physically, this means that there are different compactifications which each typically have different structures of 7-branes away from the instanton. On the instanton, though, every compactification with the same r has the same discriminant $\Delta|_{\mathcal{E}_F}$. This is a strong hint that the structure of the superpotential depends heavily on the structure of 7-branes and matter curves within the instanton worldvolume, as one might expect. We will see this in the examples we consider.

These interpretations of the physics make physical sense at a qualitative level, but still leave a number of questions unanswered. One important question is how the superpotentials of $ED3$ instantons in F-theory calculated as a function of F-theory complex structure moduli via duality with heterotic relate to standard open string considerations in type IIB. In that context, the action of a $ED3$ instanton could contain a coupling of the form $\lambda X \overline{\lambda}$, where X is a matter field and λ and $\overline{\lambda}$ are charged instanton zero modes. An instanton with such a coupling would — assuming that the structure of uncharged zero modes satisfies the appropriate conditions — generate a superpotential coupling of the form $W \sim X$. At generic points in moduli space where X has a non-zero vev, this is an uncharged superpotential correction, whereas at special points when X has zero vev it is a charged superpotential correction. $ED3$ instanton corrections in F-theory should also have such a structure, so the duals of worldsheet instanton corrections should be able to be interpreted in this way. We will show in explicit examples how at certain points in $(pfaff_\Sigma)$

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[14]Note that, though only moduli describing deformations of c appear in the Pfaffian, the structure of the polynomial is determined by the matrix structure of maps in 4.17.

there are indeed jumps in the amount of chiral matter in the theory.

4.2.1 Understanding the structure of $(pfaff_\Sigma)$

From the type II perspective, where the vanishing of the superpotential corresponds to some matter fields having zero vev, one would expect that at points in $(pfaff_\Sigma)$ there is some special gauge enhancement corresponding to non-trivial intersections of 7-branes. Recall that the $\tilde{a}_q$ are sections of a line bundle on Σ, which is the $\mathbb{P}^1$ that the instanton wraps on the heterotic side and is the intersection of the instanton with the GUT stack in F-theory. They determine the degenerations of the elliptic fiber over the instanton, and thus the positions of 7-branes in the instanton worldvolume. A natural conjecture is that gauge enhancement will often occur when the sections have common zeroes, leading to the study of resultants of these sections. In [82–84], Curio studied the structure of $(pfaff_\Sigma)$ with a number of techniques, including that of resultants. We restate some results here, and interpret in terms of $ED3$ instantons in F-theory.

As we move in the moduli space of the vector bundle, the curve $c = C \cdot \mathcal{E} \subset \mathcal{E}$ changes. We will denote the corresponding moduli space of curves as $\mathcal{M}_\mathcal{E}(c)$, and a point in this moduli space $t \in \mathcal{M}_\mathcal{E}(c)$. We will sometimes denote the corresponding curve c_t, to emphasize the dependence of c on the particular point in moduli space t. Following [82–84], we introduce the following sublocus in $\mathcal{M}_\mathcal{E}(c)$:[15]

$$\nabla = \{t \in \mathcal{M}_\mathcal{E}(c) \mid \Lambda|_{c_t} = \mathcal{O}_{c_t}\}, \tag{4.18}$$

where we have introduced

$$\Lambda = \mathcal{O}_\mathcal{E}(n\sigma|_\mathcal{E} - (r - n\chi)F). \tag{4.19}$$

In addition to Σ, we would like to define another sublocus in moduli space:

$$\mathcal{R} = \{t \in \mathcal{R} \mid \tilde{a}_n | \tilde{a}_j \quad \text{for} \quad j = 2, \ldots, n\}. \tag{4.20}$$

It was shown in [82] that $\mathcal{R} \subseteq \nabla \subset (pfaff_\Sigma)$. In fact, it is crucial in F-theory that $\tilde{a}_n$ does not divide $\tilde{a}_0$, as we will see that it would correspond to an enhancement beyond E_8, which would give a singularity that cannot be resolved while maintaining the Calabi-Yau condition. It is also worth noting that for rank 3 bundles $\mathcal{R} = \nabla$, which is the case for all four examples in [43]. (It has been further argued in [84] that $\mathcal{R} = \nabla$ in general, not just for rank 3 bundles, but we will just need the weaker $SU(3)$ result.)

What does the F-theory dual look like at points $t \in \mathcal{R}$? Taking $n = 3$ for the sake of illustration, so that the GUT group is E_6, the discriminant restricted to the instanton is

$$\Delta|_{\mathcal{E}_F} = z^8(27\tilde{a}_3^4 + z(4\tilde{a}_2^3 + 54\tilde{a}_0\tilde{a}_3^2) + z^2(12\tilde{a}_2^2 f_{4|\mathcal{E}_F} + 27\tilde{a}_0^2 + 54\tilde{a}_3^2 g_{6|\mathcal{E}_F}) + ..). \tag{4.21}$$

For $t \in \mathcal{R}$ we have that $\tilde{a}_3$ divides $\tilde{a}_2$. We can then factor out a factor of $\tilde{a}_3$ from the first

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[15]In order to avoid a clash of conventions, we have changed notation from [82–84]. The dictionary is $\nabla_{here} = \Sigma_{there}$, and $\Sigma_{here} = b_{there}$.

two terms in the expansion in z. Since in general $\tilde{a}_n$ is a section of a line bundle of degree $r - n\chi$ in $\Sigma = \mathbb{P}^1$, we have that at $r - n\chi$ points of Σ the singularity is enhanced to E_8.

This result did not really require $n = 3$, so we see that also in the case of higher rank vector bundles, for $t \in \mathcal{R}$ we find a E_8 singularity in the worldvolume of the instanton. Finally, the correspondence between an E_8 singularity in the worldvolume of the instanton and the vanishing of the superpotential also applies to lower rank groups. For a rank 1 bundle there is E_8 symmetry on the entire $\mathbb{P}^1$, and for a rank 2 bundle there is generically a point of E_8 enhancement in the instanton, since three divisors (the instanton, the E_7-brane, and a $U(1)$ flavor brane) generically intersect at a point inside a Calabi-Yau. For these ranks, there are points of E_8 enhancement in the instanton worldvolume at all points in moduli space, giving a physical reason for the fact [82] that the superpotential is identically zero for rank 1 and rank 2 bundles.

As an somewhat speculative side remark, it is perhaps worth pointing out that for F-theory GUTs with $G = SU(5)$, points of local E_8 enhancement have been advocated as giving rise to interesting phenomenological properties [85]. Nevertheless, such a point is rather non-generic in moduli space, so one may wonder why dynamics should prefer such a point (one could perhaps try to make some kind of cosmological argument along the lines of [86]). The observation here that local enhancements to E_8 symmetry are associated with vanishing non-perturbative superpotentials could perhaps be useful to give a dynamical reason why E_8 points may be preferred. It would be interesting to explore this point further.

4.3 Explicit examples

In this section we discuss the F-theory duals of explicit heterotic worldsheet instanton calculations presented in [43]. Those examples took the heterotic base to be a Hirzebruch surface $\mathbb{F}_p$, which is the toric variety specified in table 3, and wrapped a worldsheet instanton on $\Sigma = \{x = 0\}$. The Stanley-Reisner ideal is given by $SRI = \langle uv, wx \rangle$ and the

	u	v	w	x
A	1	1	p	0
Σ	0	0	1	1

Table 3. Homogeneous coordinates, divisor classes, and GLSM charges for the Hirzebruch surface $\mathbb{F}_r$.

intersection numbers are given by $A^2 = 0$, $A \cdot \Sigma = 1$, $\Sigma^2 = -p$. The canonical bundle is $K_{\mathbb{F}_p} = \mathcal{O}_{\mathbb{F}_p}(-(p+2)A - 2\Sigma) \equiv \mathcal{O}_{\mathbb{F}_p}(-(r+2), -2)$.[16]

The curve η in $B_2 = \mathbb{F}_p$ which determines the spectral cover C is given by

$$\eta = bA + (a+1)\Sigma \qquad\qquad a, b \in \mathbb{Z}. \tag{4.22}$$

---

[16] [43] uses the same coefficients, but $\mathcal{S}$ instead of Σ and $\mathcal{E}$ instead of A. We deviate from this because we have defined $\mathcal{E}$ elsewhere.

From (4.6) it can be seen that the a_q coefficients appearing in the defining equation f_C for the spectral cover are sections of

$$a_q \in H^0(\mathbb{F}_p, \mathcal{O}_{\mathbb{F}_p}(-q(p+2) + b, a + 1 - 2q)) \tag{4.23}$$

and a simple calculation, noting that $\mathcal{O}_{\mathbb{F}_p}(\Sigma)|_\Sigma = \mathcal{O}_\Sigma(-p)$ and $\mathcal{O}_{\mathbb{F}_p}(A)|_\Sigma = \mathcal{O}_\Sigma(1)$, gives

$$\tilde{a}_q \equiv a_q|_\Sigma \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(b - pa - p + q(p-2))), \tag{4.24}$$

where we have also used the fact that Σ is a $\mathbb{P}^1$, as it must be if a worldsheet instanton on it gives a superpotential correction. We write the section a_q as

$$a_q = \sum_{i=0}^{a+1-2q} x^i w^{a+1-2q-i} \sum_{j=0}^{-q(p+2)+b-p(a+1-2q-i)} a_{q;ij} \; u^j v^{-q(p+2)+b-p(a+1-2q-i)-j}. \tag{4.25}$$

The coefficients of $\tilde{a}_q$ are the vector bundle moduli that appear in $pfaff_\Sigma$, and are given by the terms in (4.25) that do not vanish when we set $x = 0$:

$$\tilde{a}_q = \sum_{i=0}^{b-pa-p+q(p-2)} a_{q;0i} \; u^i v^{b-pa-p+q(p-2)-i}. \tag{4.26}$$

Interestingly, in the case of $p = 2$ the q dependence of $\tilde{a}_q$ drops out and therefore the $\tilde{a}_q$ are all sections of the same bundle. Also, it is important to note that the sections $\tilde{a}_q$ depend only on the combination $b - pa$, rather than b and a individually. This means that two heterotic (or F-theory, via duality) compactifications which differ in a and/or b but not $b - pa$ have the same Pfaffian, $pfaff_\Sigma$. We will explain this phenomenon from an F-theory point of view.

We wish to discuss in particular F-theory duals of the examples in [43] which took V to be a rank 3 holomorphic vector bundle, so $n = 3$, and therefore from (4.9) the F-theory fourfold Y in Weierstrass form has

$$f = z^3 f_3 + z^4 f_4 \qquad g = z^4 g_4 + z^5 g_5 + z^6 g_6$$
$$\Delta = z^8 \left(27 g_4^2 + z \left(4 f_3^3 + 54 g_4 g_5 \right) + z^2 \left(27 g_5^2 + 12 f_3^2 f_4 + 54 g_4 g_6 \right) + \dots \right) \tag{4.27}$$

which can be written, via duality, in terms of the spectral surface sections a_q as

$$\Delta = z^8 \left(27 a_3^4 + z \left(4 a_2^3 + 54 a_3^2 a_0 \right) + z^2 \left(27 a_0^2 + 12 a_2^2 f_4 + 54 a_3^2 g_6 \right) + \dots \right). \tag{4.28}$$

The dots represent higher order terms in z that are not relevant for the discussion at hand.

From the structure of the discriminant, it is easy to see that there is a GUT stack with E_6 singularity at $z = 0$. The rest of the discriminant gives a divisor in B_3 which has I_1 singularity for generic moduli. The lowest order terms in z appearing in the discriminant will be the most interesting for us. For example, the intersection of $a_3 = 0$ and the GUT stack at $z = 0$ corresponds to an E_7 enhancement over a curve, and thus the possibility of

having **27**/$\overline{\mathbf{27}}$ matter localized there. We include the next order in z by defining

$$\Gamma \equiv 27\,a_3^4 + z\,(4\,a_2^3 + 54\,a_3^2 a_0) \qquad\qquad \tilde{\Gamma} \equiv \Gamma|_{x=0} = 27\,\tilde{a}_3^4 + z\,(4\,\tilde{a}_2^3 + 54\,\tilde{a}_3^2\tilde{a}_0). \quad (4.29)$$

If Γ is identically zero everywhere in B_2, then the I_1 part of the discriminant enhances to I_2 (giving $A_1 \cong SU(2)$) at $z = 0$, which (including the z^8 term) enhances to E_8 symmetry on the GUT stack. Γ does not have to be identically zero, of course, but may become zero on divisors or curves in B_3. The appearance of such divisors and/or curves at certain loci in moduli space will be important for the structure of instanton corrections.

For euclidean D-instanton corrections in weakly coupled type II, the presence of additional abelian symmetries plays an important role in determining charged superpotential corrections. In F-theory, it was suggested in [28] that the existence of curves of $SU(2)$ enhancement in the I_1 part of the discriminant is an indication of the existence of unhiggsed abelian symmetries in the compactification. The argument is essentially that chiral matter under the abelian symmetries is localized at the curve of $SU(2)$ enhancement. As one varies the moduli away from regions of $SU(2)$ enhancement the branes recombine, giving a vev to the chiral matter and higgsing the symmetry. Though the curves of $SU(2)$ enhancement that we study will often be contained inside the GUT stack, such curves generically occur away from the GUT stack and are difficult to study. This is one of the major disadvantages of the spectral cover formalism in F-theory that necessitates a more global point of view: GUT singlet matter that is chiral under abelian symmetries is generically localized away from the GUT stack. We will see that this matter can be of crucial importance for $ED3/M5$ instanton corrections.

4.3.1   A very symmetric example

We discuss a simple non-trivial example, which is example 4 of [43]. The parameters specifying the compactification are given by

$$n = 3, \qquad p = 2, \qquad b - pa = 4, \qquad \lambda = \frac{3}{2} \qquad a > 5 \qquad (4.30)$$

where the inequality for a ensures a positive spectral cover and the other parameters are as discussed previously. We emphasize again that this example includes many heterotic compactifications, since a and b are not specified, but instead only satisfy some constraints $b - pa = 4$ and $a > 5$. Thus, in each compactification, the a_q's may be sections of different bundles based on the particular values of a and b. It is only the $\tilde{a}_q$'s whose bundles are explicitly determined, which are

$$\tilde{a}_0, \tilde{a}_2, \tilde{a}_3 \in H^0(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2)). \qquad (4.31)$$

They are all sections of the same bundle because $p = 2$, which eliminates the q dependence of the bundles.

Due to the one-parameter ambiguity and a and b, there are also multiple F-theory compactifications corresponding to this example, each dual to a particular heterotic compactification specified by a choice of a and b. Each different F-theory compactification

that gives rise to the same $pfaff_\Sigma$ can have a different structure of 7-branes away from the instanton. The 7-brane structure common to all such compactifications is the structure of 7-brane over the worldvolume of the instanton, as the discriminant restricted to the instanton is the same in all of those examples, and therefore one might expect $\Delta|_{\mathcal{E}_F}$ to be crucial for the structure of $pfaff_\Sigma$.

Using the techniques reviewed in section 4.2, the Pfaffian prefactor of a superpotential correction due to a worldsheet instanton on Σ can be calculated, giving [43]

$$pfaff_\Sigma = \epsilon_{ijk} \, a_{0;0i} \, a_{2;0j} \, a_{3;0k}, \tag{4.32}$$

which is nothing but the determinant of the matrix

$$\begin{pmatrix} a_{0;00} & a_{0;01} & a_{0;02} \\ a_{1;00} & a_{1;01} & a_{1;02} \\ a_{2;00} & a_{2;01} & a_{2;02} \end{pmatrix}.$$

We have $pfaff_\Sigma$ as an explicit function of the complex structure moduli space of F-theory, and an important question is how the structure of its vanishing locus, $(pfaff_\Sigma)$ relates to 4d matter fields. At a naive glance, this connection seems like it might be possible to make because $(pfaff_\Sigma)$ is a function of complex structure moduli, which determine the structure of 7-branes and thus also of matter fields. Ultimately, we wish to relate the vanishing of $pfaff_\Sigma$ to the appearance of additional matter at certain point in moduli space.

We do this by considering particular subloci of $(pfaff_\Sigma)$. Four of the subloci and the deformations interpolating between them are displayed in figure 7. First consider the region in moduli space where $a_{0;0i}$ are turned on, but $a_{2;0i}$ and $a_{3;0i}$ are identically zero for all i. That is, $\tilde{a}_2$ and $\tilde{a}_3$ are zero. From the structure of the discriminant restricted to the instanton,

$$\Delta|_{\mathcal{E}_F} = z^8(27\tilde{a}_3^4 + z(4\tilde{a}_2^3 + 54\tilde{a}_0\tilde{a}_3^2) + z^2(12\tilde{a}_2^2 f_{4|\mathcal{E}_F} + 27\tilde{a}_0^2 + 54\tilde{a}_3^2 g_{6|\mathcal{E}_F}) + ..), \tag{4.33}$$

it is clear that such a region in moduli space corresponds to matter curves of E_8 enhancement in the instanton worldvolume. As we have said nothing about the moduli $a_{q;ij}$ for $i > 0$, which only influence the discriminant away from the instanton, this is only a curve of E_8 enhancement in Y, rather than the restoration of E_8 symmetry along the entire locus $z = 0$. The curve of E_8 involves the multiple intersections: it is located in both the instanton and the GUT stack, and also is in the locus where the I_1 part of the discriminant self intersects. It is certainly non-generic that such a curve would be present in the instanton worldvolume. From (4.32) it is straightforward to see that the Pfaffian vanishes at this locus in moduli space.

From this region with a curve of E_8 present in the instanton worldvolume, consider three simple deformations. First, turning on $a_{2;0i}$, it is clear from 4.33 that the curve of E_8 in the instanton worldvolume $\mathcal{E}_F$ becomes a curve of E_7 in $\mathcal{E}_F$. This matter curve is generically the intersection of the E_6 GUT stack with the I_1 part of the discriminant, so that $\mathbf{27}$'s or $\overline{\mathbf{27}}$'s of E_6 are present at the curve of E_7. An explicit calculation using the
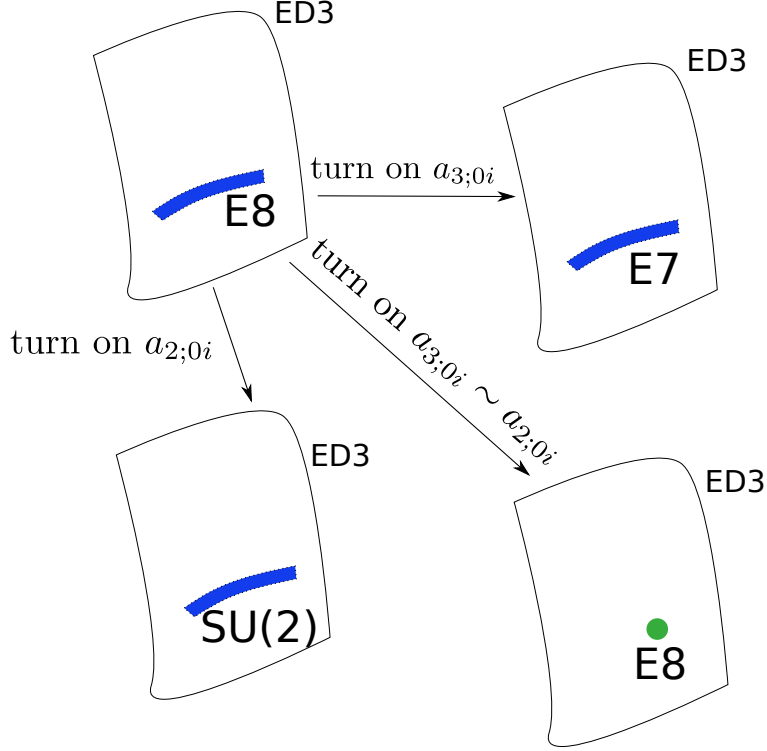
Figure 7. Depicted are three deformations from the region in moduli space with a curve of E_8 enhancement in the worldvolume of the instanton. Two correspond to deformations which leave a matter curve present in the instanton worldvolume, represented by curves of $SU(2)$ and E_7 enhancement, while a third deformation maintains a point of E_8 enhancement (generically multiple) but no matter curve is present. All four instanton configuration represent regions in moduli space where the Pfaffian (4.32) is zero.

techniques of [87] shows that this curve corresponds to a *jump* in the number of $\overline{\mathbf{27}}$'s. There is new light matter in the spectrum, which makes it not particularly surprising that the Pfaffian (4.32) vanishes at this point in moduli space.

Going back to the region of moduli space with a curve of E_8, consider instead turning on $a_{3;0i}$. In [88] it was argued that in F-theory compactifications with an E_6 GUT stack, the I_1 part of the discriminant enhances to an $I_2 = SU(2)$ singularity when $a_2 = 0$.[17] Therefore, there is a curve of $SU(2)$ enhancement in the instanton worldvolume when $\tilde{a}_2 = 0$, as is the case here. There is new light matter in the spectrum, which again makes it not particularly surprising that the Pfaffian (4.32) vanishes at this region in moduli space.

Going back to the region with a curve of E_8, turn on both $a_{2;0i}$ and $a_{3;0i}$, but so that they are proportional to one another. In this case, we are on the locus $\mathcal{R}$ in moduli space (see the discussion around (4.20)), and $\tilde{a}_3$ divides $\tilde{a}_2$. There is therefore a point of E_8 enhancement at every zero of $\tilde{a}_3$, so that (in this case) there are two points of E_8 enhancement in the worldvolume of the instanton. In this region, there is no matter curve

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[17]The conventions of [88] are slightly different than those used here, including a rescaling of one of the sections by $\frac{1}{4}$. For brevity, we do not delve into details here.

obviously present in the instanton worldvolume. Nevertheless, it is clear from (4.32) that the Pfaffian vanishes, as expected from being on $\mathcal{R}$.

Finally, we wish to mention one other locus in (pfaff_Σ) that is puzzling. If one turns off $\tilde{a}_{0,i}$ but turns on $\tilde{a}_{2,i}$ and $\tilde{a}_{3,i}$, the superpotential vanishes. However, the moduli responsible for preserving E_8 in the worldvolume of the instanton have been turned off, so that if $\tilde{a}_{2,i}$ and $\tilde{a}_{3,i}$ were also turned off, the singularity would enhance past E_8. The implications of turning off a_0 were studied in [88], with a focus on the influence of $U(1)$'s. In doing so, one leaves the regime where an 8d gauge theory localized at $z = 0$ is a good approximation. We leave the study of instantons in this region of moduli space for future work.

5   Conclusions

In this paper we have studied various aspects of the behavior of D-brane instantons in regions of strong coupling. Due to the fact that worldsheet CFT techniques and intuitions are no longer available to us, we have taken a number of complementary approaches in order to illuminate the physics.

In section 2 we gave a description of neutral zero modes valid at strong coupling. The description is based on the observation of [41] that there is anomaly inflow towards the orientifold, and it is the θ modes that cancel the anomaly on the orientifold. We reinterpreted this observation from the point of view of string junctions, which gave us a clear picture of how to describe the θ mode at strong coupling. One particularly interesting result of this analysis is that it explains how can the neutral θ mode arise out of the light, mutually non-local, charged degrees of freedom localizing in each of the components of the $O7^-$ at strong coupling.

In section 3 we turned our attention to describing the related issue of charged modes at strong coupling. In particular, we dealt with the behavior of zero modes as we move in moduli space from perturbative points ($SO(8)^4$ in our analysis) to very non-perturbative points (E_6^3). One surprising feature is that the string junctions that in one corner of moduli space give light degrees of freedom localized at each of the 7-brane stacks, in the opposite corner become very massive states, transforming in non-trivial representations of the gauge groups on the 7-brane stacks.

Finally, having obtained some understanding of how to compute the spectrum of neutral and charged zero modes at strong coupling, in section 4 we proceed to study our main object of interest: non-perturbative superpotentials in strongly coupled regimes. In particular, we give a physical analysis of the results in [42, 43], who give an exact expression for the vector bundle moduli dependence of the superpotential generated by a particular worldsheet instanton in a heterotic compactification preserving E_6 symmetry. By dualizing the configuration to F-theory, we observe that zeroes of the superpotential are related to local enhancements of the singularity of the elliptic fiber over the worldvolume of the instanton. The superpotential correction is therefore fairly independent of 7-brane behavior away from the instanton, so that the same correction could arise from many compactifications. We show that regions in moduli space where the superpotential vanishes are often

accompanied by the presence of matter curves in the worldvolume of the instanton, corresponding to the presence of additional light matter in the low energy spectrum. This agrees with what is expected from the type II perspective. We also discuss a region in moduli space where zeroes of the superpotential are accompanied by points of $E8$ enhancement in the instanton worldvolume.

There are various questions which, despite being clearly suggested by our line of attack, we have not answered in this paper, and to which we hope to come back shortly. A first question is generalizing the anomaly inflow analysis to configurations with strongly coupled stacks, such as E_6. This should provide a better understanding of the dynamics of instanton zero modes living on the stack, and clarify the mixing of charged/neutral zero modes as we move to strong coupling.

When we studied the behavior of the non-perturbative potential in section 4, we observed a clear pattern of enhancement associated with vanishing of the superpotential, at least for some loci of moduli space. Nevertheless, our analysis was not general — we do not identify the behavior of the dual F-theory background for all moduli making W vanish — and it does not give a clear F-theoretic explanation of what precisely makes W vanish. A full description of the physics will doubtlessly require some understanding of the F-theory background beyond pure geometry, in particular issues of massive $U(1)$s and G-flux will have to be addressed (see [28, 39, 40] for recent papers clarifying some of the aspects that we believe will be involved in a complete analysis).

A   Generalized Hirzebruch variety $\mathbb{F}_{pmk}$

The worldsheet instanton examples discussed in section 4.3 took $B_2 = \mathbb{F}_p$ as a base. Generically, the F-theory elliptic fibration base B_3 is a $\mathbb{P}^1$ fibration over B_2. In this case, B_2 is a Hirzebruch surface, which is itself a $\mathbb{P}^1$ fibration over $\mathbb{P}^1$. Therefore, B_3 is a $\mathbb{P}^1$ fibration over a $\mathbb{P}^1$ fibration over $\mathbb{P}^1$. This geometry is known as a *generalized Hirzebruch variety* $\mathbb{F}_{pmk}$ [77]. Parametrizing the curve $\tilde{\eta} \equiv \eta + 6\,[K_{\mathbb{F}_p}]$ as $\tilde{\eta} = m\,A + n\,\Sigma$, the GLSM charges of $\mathbb{F}_{pmk}$ are given in table 4.

We abuse notation and call two of the divisor classes in $\mathbb{F}_{pmk}$ A and Σ, which are merely pullbacks from the corresponding divisor classes in $\mathbb{F}_p$. The parameters m and k,

	u	v	w	x	y	z
A	1	1	p	0	0	m
Σ	0	0	1	1	0	k
P	0	0	0	0	1	1

Table 4. The charge matrix for the generalized Hirzebruch variety $\mathbb{F}_{pmk}$ with homogeneous coordinates u, v, w, x, y, z.

	u	v	w	x	y	z	$\tilde{x}_F$	$\tilde{y}_F$	$\tilde{z}_F$
A	1	1	p	0	0	m	$2(p+m+2)$	$3(p+m+2)$	0
Σ	0	0	1	1	0	k	$2(k+2)$	$3(k+2)$	0
P	0	0	0	0	1	1	4	6	0
σ_F	0	0	0	0	0	0	2	3	1

Table 5. The GLSM charges for a toric variety whose Calabi-Yau hypersurface is a fourfold elliptic fibration over a generalized Hirzebruch variety.

which are related to the parameters a and b in (4.22) by $a+1 = 12+6p+m$ and $b = 12+k$ determine the Chern classes of V on the heterotic side and are purely geometric on the F-theory side, describing how the $\mathbb{P}^1$ fiber of B_3 is fibered over the base B_2.

The Stanley-Reisner ideal of $\mathbb{F}_{pmk}$ is given by:

$$SRI_{\mathbb{F}_{pmk}} = \langle uv, wx, yz \rangle \qquad (A.1)$$

and the corresponding intersection ring is given by:

$$I_{\mathbb{F}_{pmk}} = P\left(A\Sigma - kAP - pEE + (kp-m)EP + (2km - k^2p)PP\right). \qquad (A.2)$$

The anticanonical class of $\mathbb{F}_{pmk}$ is given by the sum of its divisors:

$$K_{\mathbb{F}_{pmk}}^{-1} = \sum D_i = (p+m+2)A + (k+2)\Sigma + 2P. \qquad (A.3)$$

Since $\mathbb{F}_{pmk}$ is a toric variety, itself, we can construct the F-theory Calabi-Yau fourfold Y as a hypersurface in a toric variety Σ_F. This is done by taking homogeneous coordinates $(\tilde{x}_F, \tilde{y}_F, \tilde{z}_F) \in \mathbb{P}_{231}$ and choosing the GLSM charges of $\tilde{x}_F$, $\tilde{y}_F$ and $\tilde{z}_F$ so that the Weierstrass equation defines a Calabi-Yau hypersurface in Σ_F. The details of the toric variety Σ_F are given in table 5. Per usual, f and g appearing in the defining equation for Y are sections of $4\,K_{\mathbb{F}_{pmk}}^{-1}$ and $6\,K_{\mathbb{F}_{pmk}}^{-1}$, respectively. Writing the global sections f and g as a power series in z gives the parametrization of f and g discussed in section 4.3.

B 8_v untangling under $A^4BC \mapsto ACACAC$
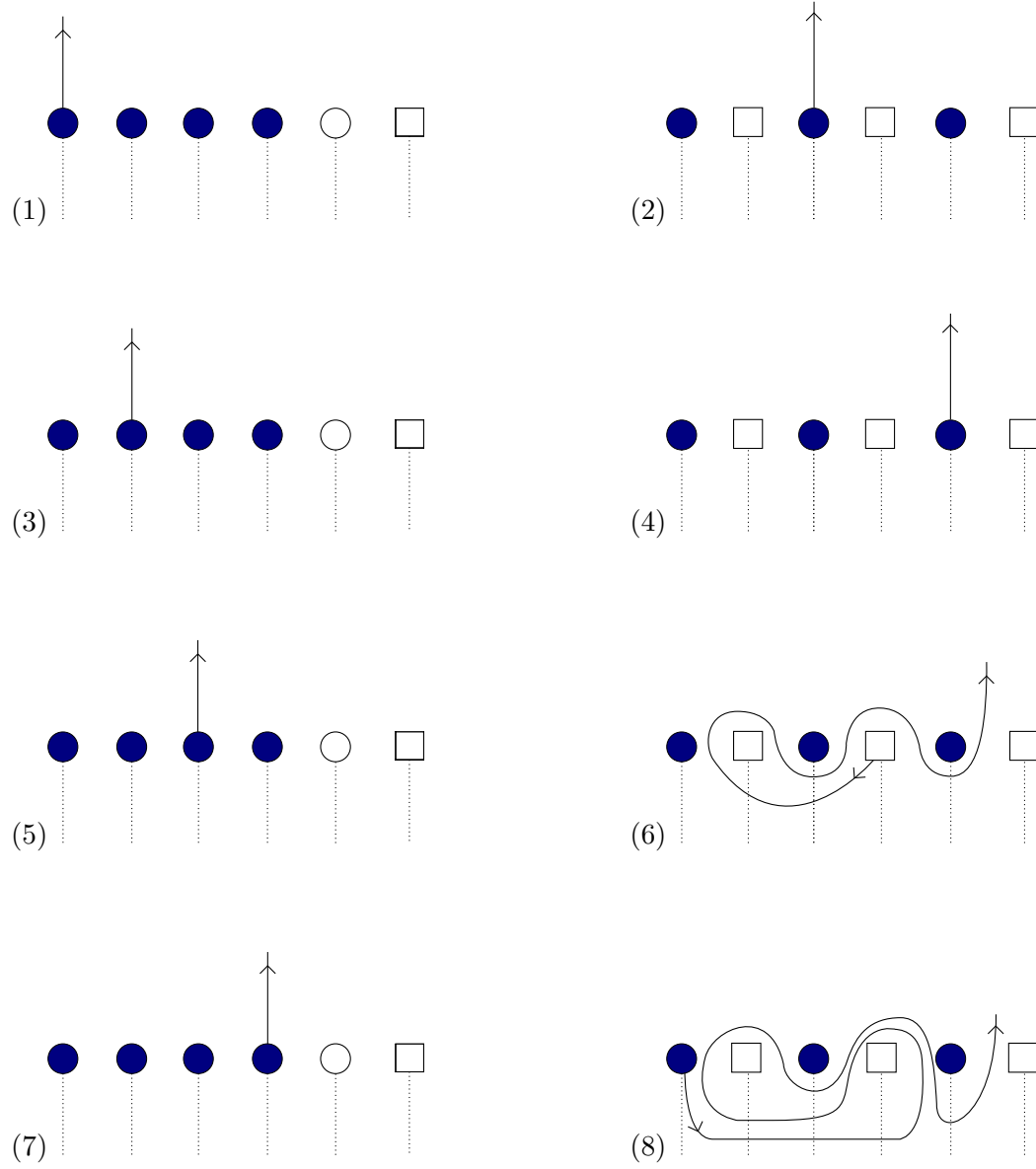
Figure 8. Depicted on the left are four strings of asymptotic charge $(1,0)$ which can be thought of as standard $ED3 - D7$ λ modes which form half of the $\mathbf{8}_v$ at points of $SO(8)$ enhancement. To right of each string configuration is a representation of the string after performing the brane motion $A^4BC \mapsto ACACAC$.
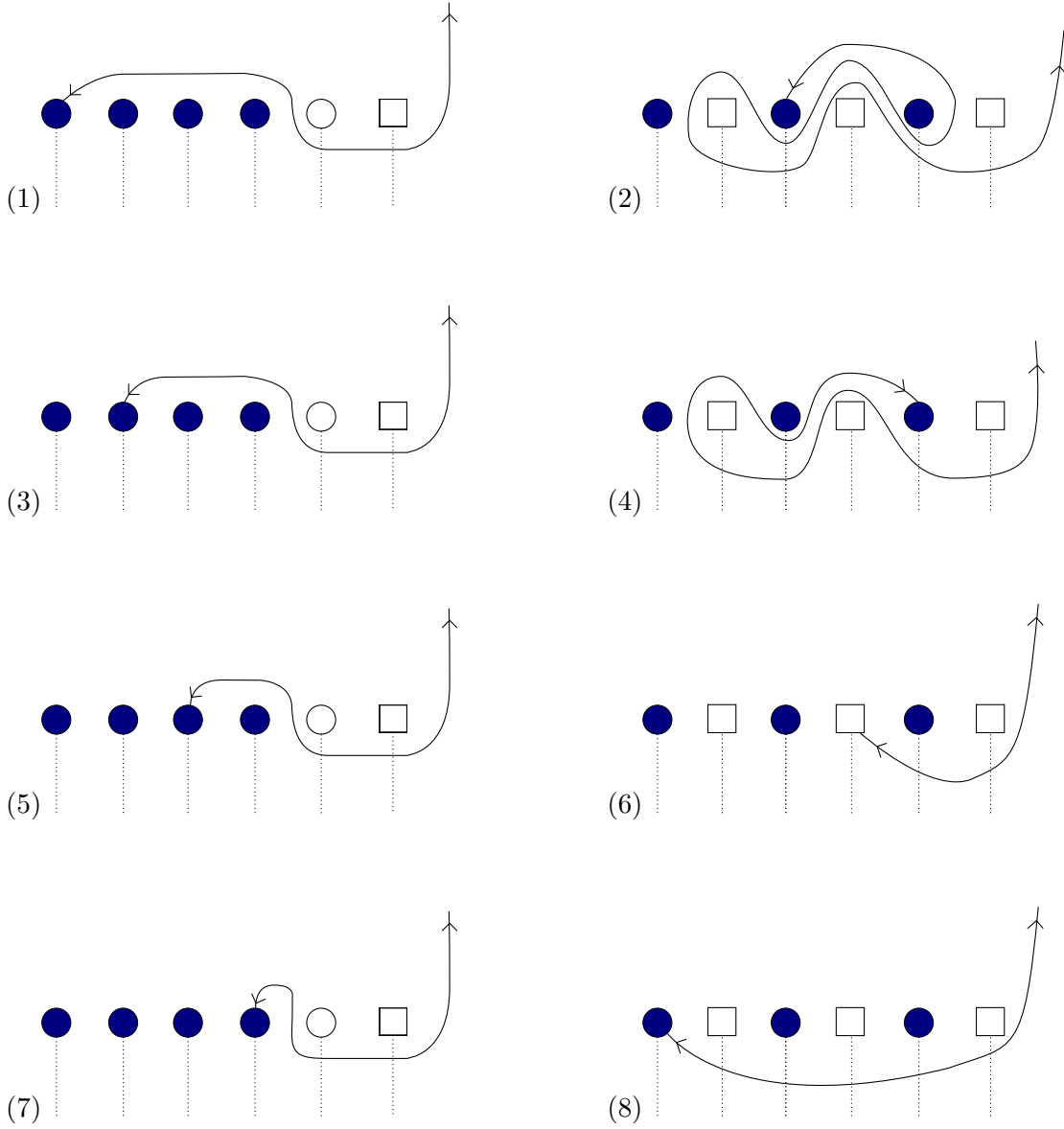
Figure 9. Depicted on the left are four λ modes whose orientations are flipped by the action of B and C, which together become an orientifold at weak coupling. They fill out the other half of the $\mathbf{8}_v$ at points of $SO(8)$ enhancement. To right of each string configuration is a representation of the string after performing the brane motion $A^4BC \mapsto ACACAC$.

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